BOSTON UNIVERSITY

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Thesis

THE USE OF IMAGINARIES IN THE SOLUTION OF REAL PROBLEMS

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I Introduction: Scope of This Work

In the following sections an attempt will be made to show how imaginaries may be applied to the solution of real problems. That the use of complex numbers as an instrument in solution is not limited to any one phase of mathematics but has a wide application will be evident from the diversity of topics covered.

An elementary knowledge of the representation of complex numbers, and operations with them will be assumed.

II DeMoivre's Theorem and Its Applications

1. De Moivre's Theorem.[1] If we write any complex number z in its trigonometric form, $z = r(\cos\theta + i\sin\theta)$, then $z^n = r^n(\cos n\theta + i\sin n\theta)$. In particular, if $r = 1$, then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

The above equation, known as De Moivre's theorem, after its discoverer, embodies one of the most famous theorems of modern analysis. It may be stated thus: The argument of the nth power of any complex number is equal to n times the argument of the number. The theorem may be shown to hold for any value of n, negative, fractional, irrational, or even imaginary, but unless n is integral $\cos n\theta + i\sin n\theta$ represents but one of the several values which $(\cos\theta + i\sin\theta)^n$ may have.

2. Sine $n\theta$ and Cosine $n\theta$ in Terms of Sine θ and Cosine θ.[2] De Moivre's theorem enables us to express the sine and cosine of any multiple angle $n\theta$ in terms of powers of the sine and cosine of θ. We need only compare separately the real parts and the imaginary parts of

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n \qquad (1)$$

after expanding the right-hand member by the binomial theorem.

For the sake of brevity, let us put

$$\cos\theta = c, \quad \sin\theta = s,$$

(1) Moritz, R. E.; Elements of Plane Trigonometry pp. [illegible].

(2) Moritz, R. E. op. cit. pp. [illegible].

the right-hand side of (1) then becomes

$$(c+is)^n = c^n + nic^{n-1}s + \frac{n(n-1)}{1\cdot 2} i^2 c^{n-2}s^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} i^3 c^{n-3}s^3$$

$$+ \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4} i^4 c^{n-4}s^4 + \text{etc.},$$

$$= c^n + nic^{n-1}s - \frac{n(n-1)}{1\cdot 2} c^{n-2}s^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} i\, c^{n-3}s^3$$

$$+ \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4} c^{n-4}s^4 + \text{etc.},$$

since $i^2=-1$, $i^3=-i$, $i^4=1$, etc.

The real part of this expression must equal cos $n\theta$, since this is the real part of the equivalent left-hand member in (1), and for a like reason the imaginary part must equal i sin $n\theta$, hence we have

$$\cos n\theta = c^n - \frac{n(n-1)}{1\cdot 2} c^{n-2}s^2 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4} c^{n-4}s^4 - \text{etc.}, \quad (2)$$

$$\sin n\theta = nc^{n-1}s - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} c^{n-3}s^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5} c^{n-5}s^5 - \text{etc.}, \quad (3)$$

For example: If $n = 4$,

$$\cos 4\theta = c^4 - \frac{4\cdot 3}{1\cdot 2} c^2s^2 + \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4} s^4 = c^4 - 6c^2s^2 + s^4$$

$$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$$

$$\sin 4\theta = 4c^3s - \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3} cs^3 = 4c^3s - 4cs^3$$

$$= 4(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta).$$

[illegible], Cosine θ and Sine θ in Terms of Sines and Cosines of Multiple Angles.(1) Since $(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = 1$, then $\cos\theta + i\sin\theta$ and $\cos\theta - i\sin\theta$ are reciprocals.

Put $z = \cos\theta + i\sin\theta$, then $\frac{1}{z} = \cos\theta - i\sin\theta$,

$z^n = \cos n\theta + i\sin n\theta$, and $\frac{1}{z^n} = \cos n\theta - i\sin n\theta$,

then $z + \frac{1}{z} = 2\cos\theta$, $\quad z - \frac{1}{z} = 2i\sin\theta$,

$z^n + \frac{1}{z^n} = 2\cos n\theta$, and $z^n - \frac{1}{z^n} = 2i\sin n\theta$.

Now let us expand $(z + \frac{1}{z})^n$ and $(z - \frac{1}{z})^n$ by the binomial theorem,

$$\left(z + \frac{1}{z}\right)^n = 2^n\cos^n\theta = z^n + nz^{n-2} + \frac{n(n-1)}{1\cdot 2}z^{n-4} + \cdots$$

$$+ \frac{n(n-1)}{1\cdot 2}\cdot\frac{1}{z^{n-4}} + n\frac{1}{z^{n-2}} + \frac{1}{z^n}, \qquad (1)$$

$$\left(z - \frac{1}{z}\right)^n = 2^n i^n\sin^n\theta = z^n - nz^{n-2} + \frac{n(n-1)}{1\cdot 2}z^{n-4} - \cdots$$

$$\pm \frac{n(n-1)}{1\cdot 2}\cdot\frac{1}{z^{n-4}} \mp n\frac{1}{z^{n-2}} \pm \frac{1}{z^n}, \qquad (2)$$

where in (2) the upper signs in the last terms are to be used when n is even and the lower signs when n is odd.

Let us group together the first and last terms in each of the expressions (1) and (2), also the second and second from the last, the third and third from the last, etc. We may then write

$$2^n\cos^n\theta = \left(z^n + \frac{1}{z^n}\right) + n\left(z^{n-2} + \frac{1}{z^{n-2}}\right) + \frac{n(n-1)}{1\cdot 2}\left(z^{n-4} + \frac{1}{z^{n-4}}\right) + \cdots, \qquad (3)$$

$$2^n i^n\sin^n\theta = \left(z^n \pm \frac{1}{z^n}\right) - n\left(z^{n-2} \pm \frac{1}{z^{n-2}}\right) + \frac{n(n-1)}{1\cdot 2}\left(z^{n-4} \pm \frac{1}{z^{n-4}}\right) + \cdots \qquad (4)$$

(1) Moritz, R. E., op. cit. pp. [illegible].

where in (4) the upper signs are to be used when n is even and the lower signs when n is odd. The total number of terms in each binomial expression is one more than the index n, and since we have grouped the terms in pairs, there will be one term left over in case n is even. This term will not contain z at all, for since the exponent of z diminishes by 2 for each successive term, it will be 0, and $z^0 = 1$.

Let us now substitute for $z^n+\frac{1}{z^n}$, $z^n-\frac{1}{z^n}$, etc., their values 2 cos $n\theta$, 2 i sin $n\theta$, etc., and divide out the common factor 2. This gives

$$2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{1\cdot 2}\cos(n-4)\theta + \text{etc.}, \qquad (5)$$

$$2^{n-1}i^n\sin^n\theta = \cos n\theta - n\cos(n-2)\theta + \frac{n(n-1)}{1\cdot 2}\cos(n-4)\theta - \text{etc.}, \qquad (6)$$

when n is even, or

$$= i\left[\sin n\theta - n\sin(n-2)\theta + \frac{n(n-1)}{1\cdot 2}\sin(n-4)\theta - \text{etc.}\right]$$

when n is odd.

The factor i disappears in either case, for when n is even, say 2m, we have $i^n = i^{2m} = (-1)^m$, which is +1 or -1 according as m is even or odd, and when n is odd say 2m+1, we can divide both sides of the equation by i and have left on the left side i^{2m} =+1 or -1 as before.

For example: If n = 4,

$$2^3\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 6$$

$$\cos^4\theta = \frac{\cos 4\theta + 4\cos 2\theta + 6}{8}$$

$$2^3 i^4\sin^4\theta = \cos 4\theta - 4\cos 2\theta + 6$$

$$\sin^4\theta = \frac{\cos 4\theta - 4\cos 2\theta + 6}{8}$$

If $n = 5$,

$$2^4\cos^5\theta = \cos 5\theta + 5\cos 3\theta + 10\cos\theta$$

$$\cos^5\theta = \frac{\cos 5\theta + 5\cos 3\theta + 10\cos\theta}{16}$$

$$2^4 i^5 \sin^5\theta = i[\sin 5\theta - 5\sin 3\theta + 10\sin\theta]$$

$$\sin^5\theta = \frac{\sin 5\theta - 5\sin 3\theta + 10\sin\theta}{16}$$

4. The nth Roots of a Complex Number.(1) Suppose it is required to find the nth root of the complex number $z = r(\cos\theta + i\sin\theta)$. Denote the root $\sqrt[n]{z}$ by $z' = r'(\cos\theta' + i\sin\theta')$. Then $(z')^n = z$, and we have by DeMoivre's theorem

$$[r'(\cos\theta' + i\sin\theta')]^n = r'^n(\cos n\theta' + i\sin n\theta') = r(\cos\theta + i\sin\theta),$$

from which

$$r' = \sqrt[n]{r},\quad n\theta' = \theta,\ \theta + 2\pi,\ \theta + 4\pi,\ \theta + 6\pi, \cdots, \theta + 2k\pi,$$

since when an angle is increased by any multiple of 2π both the sine and cosine remain unchanged. It follows that

$$r' = \sqrt[n]{r},\quad \theta' = \frac{\theta}{n},\ \frac{\theta + 2\pi}{n},\ \frac{\theta + 4\pi}{n},\ \frac{\theta + 6\pi}{n}, \cdots, \frac{\theta + k\pi}{n},$$

$$z' = \sqrt[n]{z} = r^{\frac{1}{n}}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$$

$$= r^{\frac{1}{n}}\left(\cos\frac{\theta + 2\pi}{n} + i\sin\frac{\theta + 2\pi}{n}\right)$$

$$= r^{\frac{1}{n}}\left(\cos\frac{\theta + 4\pi}{n} + i\sin\frac{\theta + 4\pi}{n}\right)$$

$$\cdots\cdots\cdots\cdots$$

$$= r^{\frac{1}{n}}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right),$$

where k is any integer.

These values are not all different, for when $k = n$ we have

$$r^{\frac{1}{n}}\left(\cos\frac{\theta + 2n\pi}{n} + i\sin\frac{\theta + 2n\pi}{n}\right) = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n} + 2\pi\right) + i\sin\left(\frac{\theta}{n} + 2\pi\right)\right]$$

$$= r^{\frac{1}{n}}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right),$$

(1) Moritz, R. E., op. cit. pp. [illegible].

and similarly

$$r^{\frac{1}{n}}\left(\cos\frac{\theta+2(n+1)\pi}{n}+i\sin\frac{\theta+2(n+1)\pi}{n}\right)=r^{\frac{1}{n}}\left(\cos\frac{\theta+2\pi}{n}+i\sin\frac{\theta+2\pi}{n}\right),$$

and generally

$$r^{\frac{1}{n}}\left(\cos\frac{\theta+2(n+\mu)\pi}{n}+i\sin\frac{\theta+2(n+\mu)\pi}{n}\right)$$
$$=r^{\frac{1}{n}}\left(\cos\frac{\theta+2\mu\pi}{n}+i\sin\frac{\theta+2\mu\pi}{n}\right),$$

so that z' has only n distinct values corresponding to the values $k = 0, 1, 2, \ldots, n-1$, that is,

<u>Every complex number $r(\cos\theta + i\sin\theta)$ has n, and only n, nth roots given by the formula,</u>

$$\underline{\left[r(\cos\theta+i\sin\theta)\right]^{\frac{1}{n}}=r^{\frac{1}{n}}\left[\cos\frac{\theta+2k\pi}{n}+i\sin\frac{\theta+2k\pi}{n}\right],}$$

<u>where k has the values $0, 1, 2, \ldots, n-1$.</u>

For example, to find the five fifth roots of 1.

In the trigonometric form $1 = \cos 0^\circ + i \sin 0^\circ$.

Hence, by the formula given above, the five roots z_0, z_1, z_2, z_3, z_4 are

$$z_0 = \cos\frac{0}{5} + i\sin\frac{0}{5}$$
$$= \cos 0^\circ + i\sin 0^\circ = 1,$$

$$z_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$$
$$= \cos 72^\circ + i\sin 72^\circ$$
$$= .3090 + .9511i,$$

$$z_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$$
$$= \cos 144^\circ + i\sin 144^\circ$$
$$= -.8090 + .5878i,$$

$$z_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$$
$$= \cos\left(2\pi - \frac{4\pi}{5}\right) + i\sin\left(2\pi - \frac{4\pi}{5}\right)$$
$$= \cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5}$$
$$= \cos 144^\circ - i\sin 144^\circ$$
$$= -.8090 - .5878i,$$

$$z_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$$

$$= \cos\left(2\pi - \frac{2\pi}{5}\right) - i\sin\left(2\pi - \frac{2\pi}{5}\right)$$

$$= \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}$$

$$= \cos 72^\circ - i\sin 72^\circ$$

$$= .3090 - .9511\, i.$$

6. Solutions of the Equation $z^n - 1 = 0$.[1] If $z^n - 1 = 0$, then $z^n = 1 = \cos 0^\circ + i\sin 0^\circ$, and the n roots are

$$z_0 = \cos\frac{0}{n} + i\sin\frac{0}{n} = 1,$$

$$z_1 = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n},$$

$$z_2 = \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n},$$

. . . .

$$z_{n-2} = \cos\frac{2(n-2)\pi}{n} + i\sin\frac{2(n-2)\pi}{n},$$

$$z_{n-1} = \cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n}.$$

Now $\cos\frac{2(n-1)\pi}{n} = \cos\left(2\pi - \frac{2\pi}{n}\right) = \cos\frac{2\pi}{n}$,

$$\sin\frac{2(n-1)\pi}{n} = \sin\left(2\pi - \frac{2\pi}{n}\right) = -\sin\frac{2\pi}{n},$$

hence $z_{n-1} = \cos\frac{2\pi}{n} - i\sin\frac{2\pi}{n}$, and similarly,

$$z_{n-2} = \cos\frac{4\pi}{n} - i\sin\frac{4\pi}{n},$$

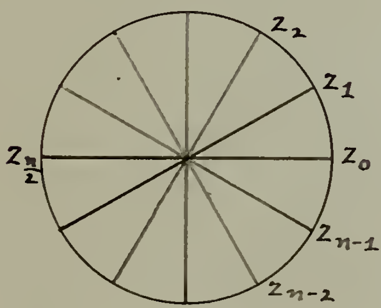
that is, z_1 and z_{n-1} are conjugate complex numbers, and likewise z_2 ~~and z~~ and z_{n-2}, z_3 and z_{n-3}, etc. are each pairs of conjugate numbers.

[1] Moritz, R. E., op. cit. pp. 1__-_3.

If $n = 2m + 1$, an odd number, then besides the first root z_0 there are m pairs of conjugate roots, that is the roots are

$$1, \cos\frac{2\pi}{n} \pm i\sin\frac{2\pi}{n}, \cos\frac{4\pi}{n} \pm i\sin\frac{4\pi}{n}, \cdots, \cos\frac{2m\pi}{n} \pm i\sin\frac{2m\pi}{n}.$$

If $n = 2m$, an even number, then $z_{\frac{n}{2}} = \cos\frac{2m\pi}{n} + i\sin\frac{2m\pi}{n}$ $= \cos\pi + i\sin\pi = -1$, and the roots are 1, $\cos\frac{2\pi}{n} \pm i\sin\frac{2\pi}{n}$, $\cos\frac{4\pi}{n} \pm i\sin\frac{4\pi}{n}, \cdots, \cos\frac{(m-1)2\pi}{n} \pm i\sin\frac{(m-1)2\pi}{n}, -1.$



Geometrically, the roots of the equation $z - 1 = 0$ are represented by the n lines (or their terminal points) drawn from the origin as center to a circle, radius unity, so that they divide the circle into n equal parts, one of these lines coinciding with the positive direction of the x-axis.

6. Solution of the Equation $z^n + 1 = 0$.(1) If $z^n + 1 = 0$, then $z^n = -1 = \cos\pi + i\sin\pi$, hence the n roots are

$$z_0 = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n},$$
$$z_1 = \cos\frac{3\pi}{n} + i\sin\frac{3\pi}{n},$$
$$z_2 = \cos\frac{5\pi}{n} + i\sin\frac{5\pi}{n},$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$z_{n-2} = \cos\frac{(2n-3)\pi}{n} + i\sin\frac{(2n-3)\pi}{n},$$
$$z_{n-1} = \cos\frac{(2n-1)\pi}{n} + i\sin\frac{(2n-1)\pi}{n},$$

where z_0 and z_{n-1}, z_1 and z_{n-2}, etc., are pairs of conjugate roots.

(1) Moritz, R. E., op. cit. pp 293-294.

If $n = 2m+1$, an odd number, then the middle root is

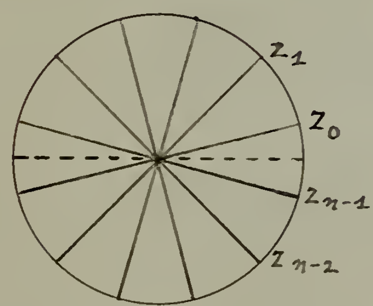
$$z_m = \cos\frac{(2m+1)\pi}{n} + i\sin\frac{(2m+1)\pi}{n} = \cos\pi + i\sin\pi = -1,$$

so that the n roots are $\cos\frac{\pi}{n} \pm i\sin\frac{\pi}{n}$, $\cos\frac{3\pi}{n} \pm i\sin\frac{3\pi}{n}$,

$$\ldots,\ \cos\frac{(2m+1)\pi}{n} \pm i\sin\frac{(2m-1)\pi}{n},\ -1.$$

If $n = 2m$, an even number, then the roots are

$$\cos\frac{\pi}{n} \pm i\sin\frac{\pi}{n},\ \cos\frac{3\pi}{n} \pm i\sin\frac{3\pi}{n},\ \ldots,\ \cos\frac{(2m-1)\pi}{n} \pm i\sin\frac{(2m-1)\pi}{n}.$$



Geometrically, the roots of the equation $z^n + 1 = 0$ are represented by the n lines (or by their terminal points) drawn from the origin as center to a circle, radius unity, so that they divide the circle into n equal parts, the positive x-axis being taken to bisect the angles between a pair of consecutive lines.

7. The Cube Roots of Unity.[1] Let u_0, u_1, u_2 represent the three cube roots of 1, then by section 4

$$u_0 = \cos 0 + i\sin 0 = 1,$$

$$u_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \frac{-1}{2} + i\frac{\sqrt{3}}{2},$$

$$u_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \frac{-1}{2} - i\frac{\sqrt{3}}{2}.$$

Now $u_1^2 = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = u_2,$

$$u_2^2 = \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)^2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = u_1,$$

also $u_1 u_2 = u_1^3 = u_2^3 = 1$, that is

The square of either of the imaginary cube roots of unity equals the other, and their product equals unity.

[1] Moritz, R. E., op. cit. p. 295.

We may then denote the cube roots of unity by 1, ω, ω^2, where ω is either one of the roots u_1, u_2.

8. <u>The Cube Roots of Any Number</u>.(1) Let $r(\cos\theta + i\sin\theta)$ represent any number and z_0, z_1, z_2 its cube roots. We have

$$z_0 = r^{\frac{1}{3}}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right),$$

$$z_1 = r^{\frac{1}{3}}\left(\cos\frac{\theta+2\pi}{3} + i\sin\frac{\theta+2\pi}{3}\right)$$

$$= r^{\frac{1}{3}}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right)\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = \omega z_0,$$

$$z_2 = r^{\frac{1}{3}}\left(\cos\frac{\theta+4\pi}{3} + i\sin\frac{\theta+4\pi}{3}\right)$$

$$= r^{\frac{1}{3}}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right)\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = \omega^2 z_0,$$

where ω has the meaning given in the preceding section.

Furthermore, by applying the results just obtained

$$\omega z_1 = \omega^2 z_0 = z_2, \qquad \omega^2 z_1 = \omega^3 z_0 = z_0,$$

$$\omega z_2 = \omega^3 z_0 = z_0, \qquad \omega^2 z_2 = \omega^4 z_0 = \omega z_0 = z_1.$$

Hence, <u>it appears that any two cube roots of a number may be obtained by multiplying the third by ω and ω^2, respectively, where $\omega = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$</u>.

9. <u>The Solution of Cubic Equations</u>.(2) Every cubic equation can be expressed in the form

$$a_0 x^3 + 3a_1 x^2 + 3a_2 x + a_3 = 0 \qquad (1)$$

where a_0, a_1, a_2, a_3 are given numbers.

(1) Moritz, R. E., op. cit. p. [illegible].
(2) Moritz, R. E., op. cit. pp. [illegible]-29.

Equation (1) can be replaced by another in which the second term is missing by putting

$$x = \frac{z - a_1}{a_0} \tag{2}$$

On making this substitution, equation (1) reduces to

$$z^3 + 3Hz + G = 0 \tag{3}$$

where $H = a_0 a_2 - a_1^2$, $G = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3$. (4)

Now put

$$z = u + v, \tag{5}$$

then (3) becomes

$$(u+v)^3 + 3H(u+v) + G = u^3 + v^3 + 3(u+v)(H+uv) + G = 0 \tag{6}$$

If furthermore we put

$$H + uv = 0, \text{ that is, } \quad uv = -H \tag{7}$$

then (6) becomes

$$u^3 + v^3 = -G, \tag{8}$$

and substituting for v in (8) its values from (7)

$$u^3 - \frac{H^3}{u^3} = -G, \text{ or } u^6 + Gu^3 - H^3 = 0, \tag{9}$$

(9) is a quadratic equation in u^3. Solving

$$u^3 = \frac{-G \pm \sqrt{G^2 + 4H^3}}{2}, \qquad v^3 = -G - u^3 = \frac{-G \mp \sqrt{G^2 + 4H^3}}{2},$$

so that for either sign we obtain

$$z = u + v = \sqrt[3]{\frac{-G + \sqrt{G^2 + 4H^3}}{2}} + \sqrt[3]{\frac{-G - \sqrt{G^2 + 4H^3}}{2}} \tag{10}$$

From (7) $v = \frac{-H}{u}$ so that (10) may be written $z = u - \frac{H}{u}$,

where $u = \sqrt[3]{\frac{-G + \sqrt{G^2 + 4H^3}}{2}}$.

But the cube root of every number has three values, which by section 8 may be denoted by u, ωu, $\omega^2 u$, respectively, where u is any one of these roots. The three values of z which satisfy the equation (3) are therefore

$$\begin{aligned} z_0 &= u - \frac{H}{u}, \\ z_1 &= \omega u - \frac{H}{\omega u} = \omega u - \frac{\omega^2 H}{u}, \\ z_2 &= \omega^2 u - \frac{H}{\omega^2 u} = \omega^2 u - \frac{\omega H}{u}, \end{aligned} \tag{11}$$

where u is either one of the cube roots of

$$\frac{-G + \sqrt{G^2 + 4H^3}}{2}. \tag{12}$$

Whence

$$x = \frac{z - a_1}{a_2}. \tag{13}$$

In applying this method to the solution of any equation of the form (1), we first find H and G from (4), then u from (12), then the three values of z from (11) and finally the three corresponding values of x from (13).

For example, to find the roots of the equation

$$8x^3 + 12x^2 - 42x - 95 = 0.$$

$$H = 8(-14) - 4^2 = -128. \qquad G = 8^2(-95) - 3(8)(4)(-14) + 2(4)^3 = -4608.$$

$$u = \sqrt[3]{\frac{4608 + \sqrt{21,233,664 - 8,388,608}}{2}}$$

$$= \sqrt[3]{\frac{4608 + \sqrt{12,845,056}}{2}}$$

$$= \sqrt[3]{\frac{4608 + 3584}{2}} = \sqrt[3]{4096} = 16.$$

$$z = 16 + \frac{128}{16}$$

$$= 16 + 8 = \underline{24}.$$

$$z_1 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)16 + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)8$$

$$= -8 + 8i\sqrt{3} - 4 - 4i\sqrt{3}$$

$$= \underline{-12 + 4i\sqrt{3}}.$$

$$z_2 = \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)16 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)8$$

$$= -8 - 8i\sqrt{3} - 4 + 4i\sqrt{3}$$

$$= \underline{-12 - 4i\sqrt{3}}.$$

$$x_0 = \frac{24-4}{8} = \underline{2.5}.$$

$$x_1 = \frac{-12 + 4i\sqrt{3} - 4}{8} = \underline{-2 + \frac{i\sqrt{3}}{2}}.$$

$$x_2 = \frac{-12 - 4i\sqrt{3} - 4}{8} = \underline{-2 - \frac{i\sqrt{3}}{2}}.$$

10. <u>The Irreducible Case of the Cubic Equation.</u>[1] When $G^2 + 4H^3$ is positive, as in the preceding example, its square root is real, and u, which is the cube root of $\frac{-G + \sqrt{G^2 + 4H^3}}{2}$ can be found by the rules of arithmetic. But if $G^2 + 4H^3$ is negative, its square root will be imaginary, and we must employ DeMoivre's theorem to find u. This is the so-called "irreducible case" of the cubic equation. Its solution is as follows:

Since $G^2 + 4H^3$ is negative, $-(G^2 + 4H^3)$ will be positive, and we may put

$$u^3 = \frac{-G + i\sqrt{-(G^2 + 4H^3)}}{2} = r(\cos\theta + i\sin\theta),$$

(1) Moritz, R. E., op. cit. --, [illegible].

Whence $r = \sqrt{\dfrac{(-G)^2 + [-(G^2+4H^3)]}{4}}$ (1) $= \sqrt{-H^3}$,

$$\therefore \cos\theta = \frac{-G}{2\sqrt{-H^3}} \text{ (2)}.$$

$$u = r^{\frac{1}{3}}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right) = \sqrt{-H}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right),$$

$$-\frac{H}{u} = \sqrt{-H}\left(\cos\frac{\theta}{3} - i\sin\frac{\theta}{3}\right),$$

$$\omega u = \sqrt{-H}\left(\cos\frac{\theta+2\pi}{3} + i\sin\frac{\theta+2\pi}{3}\right),$$

$$\frac{-H}{\omega u} = \sqrt{-H}\left(\cos\frac{\theta+2\pi}{3} - i\sin\frac{\theta+2\pi}{3}\right),$$

$$\omega^2 u = \sqrt{-H}\left(\cos\frac{\theta+4\pi}{3} + i\sin\frac{\theta+4\pi}{3}\right),$$

$$\frac{-H}{\omega^2 u} = \sqrt{-H}\left(\cos\frac{\theta+4\pi}{3} - i\sin\frac{\theta+4\pi}{3}\right).$$

Hence, $z_0 = u - \dfrac{H}{u} = 2\sqrt{-H}\,\cos\dfrac{\theta}{3}$,

$$z_1 = \omega u - \frac{H}{\omega u} = 2\sqrt{-H}\,\cos\frac{\theta+2\pi}{3},$$

$$z_2 = \omega^2 u - \frac{H}{\omega^2 u} = 2\sqrt{-H}\,\cos\frac{\theta+4\pi}{3},$$

and

$$x = \frac{z - a_1}{a_0}.$$

For example, to find the roots of $x^3 + 9x^2 + 24x + 19 = 0$.

$H = 1(8) - 3^2 = -1$. $G = 1(19) - 3(1)(3)(8) + 2(3)^3 = +1$.

$G^2 + 4H^3 = 1 - 4 = -3$, hence, we have an irreducible case.

$$\cos\theta = \frac{-G}{2\sqrt{-H^3}} = \frac{-1}{2(1)} = -\frac{1}{2} = -.5000, \quad \theta = 120^\circ.$$

$$z_0 = 2\sqrt{-H}\,\cos\frac{\theta}{3} = 2\cos 40^\circ = 2(.7660) = 1.5320.$$

(1) $r = \sqrt{x^2 + y^2}$.

(2) $\cos\theta = \dfrac{x}{r}$.

$$z_1 = 2\sqrt{-H}\cos\frac{\theta+2\pi}{3} = 2\cos 160^\circ = 2(-.9397) = -1.8794.$$
$$z_2 = 2\sqrt{-H}\cos\frac{\theta+4\pi}{3} = 2\cos 280^\circ = 2(.1736) = .3472.$$
$$x_0 = \frac{z_0 - a_1}{a_0} = \frac{1.5320-3}{1} = \underline{-1.4680.}$$
$$x_1 = \frac{z_1 - a_1}{a_0} = \frac{-1.8794-3}{1} = \underline{-4.8794.}$$
$$x_2 = \frac{z_2 - a_1}{a_0} = \frac{.3472-3}{1} = \underline{-2.6528.}$$

11. <u>The Regular Polygon of 17 Sides.</u>(1) Since the rational operations of addition, subtraction, multiplication, and division, and irrational operations involving only square roots can be constructed with straight edge and compasses and, since every individual geometrical construction which can be reduced to the intersection of two lines, a straight line and a circle, or two circles, is equivalent to a rational operation, or the extraction of a square root, we may state the following theorem. <u>The necessary and sufficient condition that an analytic expression can be constructed with straight edge and compasses is that it can be derived from the known quantities by a finite number of rational operations and square roots.</u>

Gauss proved that the division of a circle into n equal parts was possible for a prime number p when, and only when, $p = 2^{2^\mu}+1$.(2) If $\mu = 2$, then $p = 17$.

Let us trace in the z-plane $(z = x + iy)$ a circle of radius 1. To divide this circle into 17 equal parts, beginning at $z = 1$, is the same as to solve the equation $z^{17} - 1 = 0$.(3) This equation has the root $z = 1$. Let us suppress this root by

(1) Beman and Smith: Translation of Klein's Famous Problems of Elementary Geometry, p. 3, 16, 19, 24 - 32.
(2) see Gauss: Disquisitiones Arithmeticae, or Beman and Smith: Famous Problems of Elementary Geometry, Chap. III.
(3) see section 5.

dividing by $z - 1$, which is the same geometrically as to disregard the initial point of the division. We thus obtain the equation $z^{16} + z^{15} + z^{14} + \dots\dots\dots + z^2 + z + 1 = 0$, which we will call the cyclotomic equation. We shall show that we can express its roots in terms of square roots.

We know that the roots can be put into transcendental form

$$e_k = \cos\frac{2k\pi}{17} + i\sin\frac{2k\pi}{17}\ (k = 1, 2, \cdots\cdots, 16);$$

and if $e_1 = \cos\frac{2\pi}{17} + i\sin\frac{2\pi}{17}$, that $e_k = e_1^k$. The selection of e_1 is arbitrary, but for the construction it is essential to indicate some e as the point of departure. Having fixed upon e_1, the angle corresponding to e_k is k times the angle corresponding to e_1, which completely determines e_k.

To arrange the 16 roots of the equation in a cycle in a determinate order we shall make use of the theory of congruences,(1) and form a primitive root to the modulus 17. A number, a, is by definition a primitive root to the modulus 17 when $a^s \equiv 1 \pmod{17}$ has for least solution $s = 17 - 1 = 16$, where a is an integer not divisible by p, and s is the least exponent which satisfies the congruence.

The number 3 possesses this property;(2) for we have

$3^1 \equiv 3 \pmod{17}$	$3^5 \equiv 5 \pmod{17}$	$3^9 \equiv 14 \pmod{17}$	$3^{13} \equiv 12 \pmod{17}$
$3^2 \equiv 9 \pmod{17}$	$3^6 \equiv 15 \pmod{17}$	$3^{10} \equiv 8 \pmod{17}$	$3^{14} \equiv 2 \pmod{17}$
$3^3 \equiv 10 \pmod{17}$	$3^7 \equiv 11 \pmod{17}$	$3^{11} \equiv 7 \pmod{17}$	$3^{15} \equiv 6 \pmod{17}$
$3^4 \equiv 13 \pmod{17}$	$3^8 \equiv 16 \pmod{17}$	$3^{12} \equiv 4 \pmod{17}$	$3^{16} \equiv 1 \pmod{17}$.

(1) If a and b are any two integers, positive, or zero, or negative, whose difference is divisible by m, a and b are said to be congruent according to the modulus m (written $a \equiv b \pmod{m}$). see Carmichael; The Theory of Numbers, p. 37ff.

(2) Any prime integer which is not a factor or multiple of 17 - 1 will form a primitive root for a; 3 is selected because it is the smallest integer which satisfies the condition.

Let us then arrange the roots e_k so that their indices are the preceding remainders in order

$$e_3, e_9, e_{10}, e_{13}, e_5, e_{15}, e_{11}, e_{16}, e_{14}, e_8, e_7, e_4, e_{12}, e_2, e_6, e_1 .$$

Notice that if r is the remainder of 3^k (mod 17), we have

$$3^k = 17q + r,$$

whence $e_r = e_1^r = e_1^{3^k}$.

If r' is the next remainder, we have similarly

$$e_{r'} = e_1^{3^{k+1}} = \left(e_1^{3^k}\right)^3 = (e_r)^3 .$$

Hence, in this series each root is the cube of the preceding.

Gauss's method consists in decomposing this cycle into sums containing 8, 4, 2, 1 roots respectively, corresponding to the divisors of 16. Each of these sums is called a period. The periods thus obtained may be calculated successively as the roots of certain quadratic equations.

The process just outlined is only a particular case of that employed in the general case of the division into p equal parts. The $p-1$ roots of the cyclotomic equations are cyclically arranged by means of a primitive root of p, and the periods may be calculated as roots of certain auxiliary equations. The degree of these last depends upon the prime factors of $p-1$. They are not necessarily equations of the second degree.

In our case of the 17 roots the periods may be formed in the following manner: Form two periods of 8 roots by taking in the cycle, first, the roots of even order, then those of

odd order. Designate these periods by x_1 and x_2, and replace each root by its index. We may then write symbolically

$$x_1 = 9 + 13 + 15 + 16 + 8 + 4 + 2 + 1,$$

$$x_2 = 3 + 10 + 5 + 11 + 14 + 7 + 12 + 6.$$

Operating upon x_1 and x_2 in the same way, we form 4 periods of 4 terms:

$$y_1 = 13 + 16 + 4 + 1,$$

$$y_2 = 9 + 15 + 8 + 2,$$

$$y_3 = 10 + 11 + 7 + 6,$$

$$y_4 = 3 + 5 + 14 + 12.$$

Operating in the same way upon the y's, we obtain 8 periods of 2 terms:

$z_1 = 16 + 1$	$z_5 = 11 + 6$
$z_2 = 13 + 4$	$z_6 = 10 + 7$
$z_3 = 15 + 2$	$z_7 = 5 + 12$
$z_4 = 9 + 8$	$z_8 = 3 + 14$

It is readily seen that the sum of the remainders corresponding to the roots forming a period z is always equal to 17. These roots are then e_r and e_{17-r}.

$$e_r = \cos r\frac{2\pi}{17} + i \sin r\frac{2\pi}{17},$$

$$e_{r'} = e_{17-r} = \cos(17-r)\frac{2\pi}{17} + i \sin(17-r)\frac{2\pi}{17}$$

$$= \cos r\frac{2\pi}{17} - i \sin r\frac{2\pi}{17}$$

Therefore all the periods z are real, and we readily obtain

$$z_1 = 2\cos\frac{2\pi}{17}, \qquad z_5 = 2\cos(6)\frac{2\pi}{17};$$
$$z_2 = 2\cos(4)\frac{2\pi}{17}, \qquad z_6 = 2\cos(7)\frac{2\pi}{17},$$
$$z_3 = 2\cos(2)\frac{2\pi}{17}, \qquad z_7 = 2\cos(5)\frac{2\pi}{17},$$
$$z_4 = 2\cos(8)\frac{2\pi}{17}, \qquad z_8 = 2\cos(3)\frac{2\pi}{17}.$$

Moreover, by definition,

$$x_1 = z_1 + z_2 + z_3 + z_4, \qquad x_2 = z_5 + z_6 + z_7 + z_8,$$
$$y_1 = z_1 + z_2, \quad y_2 = z_3 + z_4, \qquad y_3 = z_5 + z_6, \quad y_4 = z_7 + z_8.$$

It will be necessary to determine the relative magnitude of the different periods. For this purpose we shall employ the following artifice: We shall divide the semicircle of unit radius into 17 equal parts (see fig. 1) and denote by $s_1, s_2, \ldots, s_{17}$ the distances of the consecutive points of division $A_1, A_2, \ldots, A_{17}$ from the initial point of the semicircle, s_{17} being equal to the diameter, i.e., equal to 2. The angle $A_k A_{17} O$ has the same measure as half the arc $A_k O$ which equals $\frac{2k\pi}{34}$. Hence $s_k = 2\sin\frac{k\pi}{34} = 2\cos\frac{(17-k)\pi}{34}$.

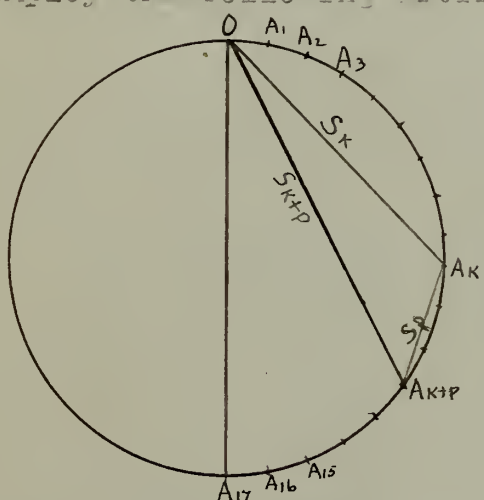


Fig 1.

That this right member may be identical with $2\cos(h)\frac{2\pi}{17}$, we must have $4h = 17 - k$, or $k = 17 - 4h$. Giving to h the values 1, 2, 3, 4, 5, 6, 7, 8 we find for k the values 13, 9, 5, 1, −3, −7, −11, −15.

Hence,

$$z_1 = s_{13}, \qquad z_5 = -s_7,$$
$$z_2 = s_1, \qquad z_6 = -s_{11},$$
$$z_3 = s_9, \qquad z_7 = -s_3,$$
$$z_4 = -s_{15}, \qquad z_8 = s_5,$$

The figure shows that s_k increases with the index; hence the order of increasing magnitude of the periods z is

$$z_4, z_6, z_5, z_7, z_2, z_8, z_3, z_1.$$

Moreover, the chord $A_k A_{k+p}$ subtends p divisions of the semicircumference and is equal to s_p. The triangle $OA_k A_{k+p}$ shows that $s_{k+p} < s_k + s_p$, and, a fortiori,

$$s_{k+p} < s_{k+r} + s_{p+r'}.$$

Calculating the differences two and two of the periods y, we find

$$y_1 - y_2 = s_{13} + s_1 - s_9 + s_{15} > 0,$$
$$y_1 - y_3 = s_{13} + s_1 + s_7 + s_{11} > 0,$$
$$y_1 - y_4 = s_{13} + s_1 + s_3 - s_5 > 0,$$
$$y_2 - y_3 = s_9 - s_{15} + s_7 + s_{11} > 0,$$
$$y_2 - y_4 = s_9 - s_{15} + s_3 - s_5 < 0,$$
$$y_3 - y_4 = -s_7 - s_{11} + s_3 - s_5 < 0.$$

Hence, $y_3 < y_2 < y_4 < y_1$,

and $x_2 < x_1$.

We now propose to calculate $z_1 = 2\cos\frac{2\pi}{17}$. After making this calculation and constructing z_1, we can easily deduce the side of the regular polygon of 17 sides. In order to find the

quadratic equation satisfied by the periods, we proceed to determine symmetric functions of the periods.

Associating z_1 with the period z_2 and thus forming the period y_1, we have first,

$$z_1 + z_2 = y_1 .$$

Let us now determine $z_1 z_2$. We have $z_1 z_2 = (1 + 16)(13 + 4)$, where the symbolic product kp represents

$$e_k \cdot e_p = e_{k+p} .$$

Hence it should be represented symbolically by $k+p$, remembering to subtract 17 from $k+p$ as often as possible.

Thus, $$z_1 z_2 = 12 + 3 + 14 + 5 = y_4 .$$

Therefore z_1 and z_2 are the roots of the quadratic equation,

$$z^2 - y_1 z + y_4 = 0 \tag{1}$$

whence, since $z_1 > z_2$, $z_1 = \dfrac{y_1 + \sqrt{y_1^2 - 4y_4}}{2}$, $z_2 = \dfrac{y_1 - \sqrt{y_1^2 - 4y_4}}{2}$.

We must now determine y_1 and y_4. Associating y_1 with the period y_2, thus forming the period x_1, and y_3 with the period y_4, thus forming the period x_2, we have, first $y_1 + y_2 = x_1$.

Then $y_1 y_2 = (13 + 16 + 4 + 1)(9 + 15 + 8 + 2)$.

Expanding symbolically,(1) the second member becomes equal to the sum of all the roots; that is, to -1.(2) Therefore, y_1 and y_2 are the roots of the equation

$$y^2 - x_1 y - 1 = 0 , \tag{2}$$

whence, since $y_1 > y_2$, $y_1 = \dfrac{x_1 + \sqrt{x_1^2 + 4}}{2}$, $y_2 = \dfrac{x_1 - \sqrt{x_1^2 + 4}}{2}$.

(1) $(13+16+4+1)(9+15+8+2) = 5+11+4+15+8+14+7+1+13+2+12+6+10+16+9+3$.

(2) See Reitz and Crathorne: Introductory College Algebra, p. 30[illegible].

Similarly, $y_3 + y_4 = x_2$, and $y_3 y_4 = -1$.

Hence y_3 and y_4 are the roots of the equation,

$$y^2 - x_2 y - 1 = 0; \tag{3}$$

whence, since $y_4 > y_3$, $y_4 = \dfrac{x_2 + \sqrt{x_2^2+4}}{2}$, $y_3 = \dfrac{x_2 - \sqrt{x_2^2+4}}{2}$.

It now remains to determine x_1 and x_2. Since $x_1 + x_2$ is equal to the sum of all the roots,

$$x_1 + x_2 = -1.$$

Further, $x_1 x_2 = (13+16+4+1+9+15+8+2)(10+11+7+6+3+5+14+12)$.

Expanding symbolically,[1] each root occurs 4 times, and thus

$$x_1 x_2 = -4.$$

Therefore, x_1 and x_2 are roots of the quadratic equation

$$x^2 + x - 4 = 0; \tag{4}$$

hence, since $x_1 > x_2$, $x_1 = \dfrac{-1+\sqrt{17}}{2}$, $x_2 = \dfrac{-1-\sqrt{17}}{2}$.

Solving equations (4), (3), (2), (1) in succession, z_1 is determined by a series of square roots.

Effecting the calculations, we see that z_1 depends upon the four square roots $\sqrt{17}$, $\sqrt{x_1^2+4}$, $\sqrt{x_2^2+4}$, $\sqrt{y_1^2 - 4y_4}$.

If we wish to reduce z_1 to the normal form[2] we must see whether any one of these square roots can be expressed rationally in terms of the others.

Now from the roots of (3),

$$\sqrt{x_1^2+4} = y_1 - y_2, \qquad \sqrt{x_2^2+4} = y_4 - y_3.$$

(1) (13+16+4+1+9+15+8+2)(10+11+7+6+3+5+14+12)
= 6+7+3+2+16+1+10+8+9+10+6+5+2+4+13+11
+14+15+11+10+7+9+1+16+11+12+8+7+4+6+15+13
+2+3+16+15+12+14+6+4+8+9+5+4+1+3+12+10
+1+2+15+14+11+13+5+3+12+13+9+8+5+7+16+14
= 4(6+7+3+2+16+1+10+8+9+5+4+13+11+14+15+12).

(2) See Beman and Smith: op. cit. pp. 6-7.

Expanding symbolically,(1) we verify that

$$(y_1 - y_2)(y_4 - y_3) = 2(x_1 - x_2),$$

that is, $\sqrt{x_1^2+4}\ \sqrt{x_2^2+4} = 2\sqrt{17}$.

Hence $\sqrt{x_2^2+4}$ can be expressed rationally in terms of the other two square roots. This equation shows that if two of the three differences $y_1 - y_2$, $y_4 - y_3$, $x_1 - x_2$ are positive, the same is true of the third, which agrees with the results obtained directly.

Replacing now x_1, y_1, y_4 by their numerical values, we obtain in succession

$$x_1 = \frac{-1+\sqrt{17}}{2},$$

$$y_1 = \frac{-1+\sqrt{17}+\sqrt{34-2\sqrt{17}}}{4},$$

$$y_4 = \frac{-1-\sqrt{17}+\sqrt{34+2\sqrt{17}}}{4}$$

$$z_1 = \frac{-1+\sqrt{17}+\sqrt{34-2\sqrt{17}}}{8}+\frac{\sqrt{68+12\sqrt{17}-16\sqrt{34+2\sqrt{17}}-2(1-\sqrt{17})\sqrt{34-2\sqrt{17}}}}{8}.$$

If $\frac{z_1}{2}$ is laid off from the origin on the axis of reals, and a perpendicular erected at its extremity, the intersections of this perpendicular with the unit circle will determine the second and sixteenth vertices of the required polygon.

Therefore, we can construct a regular polygon of 17 sides with straightedge and compass since we have shown that it can be derived from known quantities by a finite number of rational operations and square roots.(2)

(1) $(y_1 - y_2)(y_4 - y_3) = ($[illegible]$)($[illegible]$)$
$= $ [illegible]
$= $ [illegible]
$= $ [illegible]
$= 2(x_1 - x_2)$.

(2) For the actual construction see [illegible] and Smith op. cit., pp. [illegible].

III Euler's Theorem, $e^{i\theta} = \cos\theta + i \sin\theta$.

1. First Proof, Using Infinite Series.(1) The following three definitions may be made, since each of these three series is convergent.(2)

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\cdots \tag{1}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\cdots \tag{2}$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\cdots + \frac{z^n}{n!} + \cdots\cdots \tag{3}$$

If in (3), we put $z = i\theta$, then

$$e^{i\theta} = 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \cdots\cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \cdots\cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta + i \sin\theta .$$

This result known as Euler's theorem, is next to DeMoivre's theorem one of the most important results of modern analysis. Since every number, real or complex, can be expressed in the form $r(\cos\theta + i \sin\theta)$, the theorem may be stated thus:

<u>Every number may be expressed in the form $re^{i\theta}$, where r is the modulus and θ the argument of the number in its trigonometric form.</u>

2. Second Proof, Using DeMoivre's Theorem.(3) Euler's theorem may be established by the aid of DeMoivre's theorem as follows.

(1) Moritz, R. E., op. cit. pp [illegible].
(2) See Moritz, op. cit. §1[illegible] ([illegible]) p. 31[illegible], §§1[illegible], 1[illegible]0 pp. [illegible].
(3) Moritz, R. E. op. cit. p. [illegible].

By DeMoivre's theorem

$$\cos\theta + i\sin\theta = \left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)^n, \qquad \text{for every value of } n.$$

Consider θ constant, and let n increase without limit. Then $\sin\frac{\theta}{n}$ approaches $\frac{\theta}{n}$, $\cos\frac{\theta}{n}$ approaches 1, that is $\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)^n$ approaches the same limit as $\left(1+\frac{i\theta}{n}\right)^n$.

Put $\frac{i\theta}{n} = \frac{1}{m}$, that is, $n = mi\theta$, then

$$\left(1+\frac{i\theta}{n}\right)^n = \left(1+\frac{1}{m}\right)^{mi\theta} = \left[\left(1+\frac{1}{m}\right)^m\right]^{i\theta}.$$

As n increases without limit, m also increases without limit, but the limit of $\left(1+\frac{1}{m}\right)^m$ as m increases without limit, is e,[(1)] hence the limit of $\left(1+\frac{i\theta}{n}\right)^n$ as n increases without limit is $e^{i\theta}$, and we have

$$\underline{\cos\theta + i\sin\theta = e^{i\theta}.}$$

3. Geometrical Representation of Euler's Theorem.[(2)] From the following geometrical construction we may see that the limit of $\left(1+\frac{i\theta}{n}\right)^n$, as n increases without limit is $\cos\theta + i\sin\theta$. Let AOQ be any angle θ, AQ the subtended arc drawn with a unit radius $AO = 1$. At A construct $AP = \text{arc } AQ = \theta$ perpendicular to OA, and let AP_1, be one of the n equal parts of AP. Then $AP_1 = \frac{\theta}{n}$.

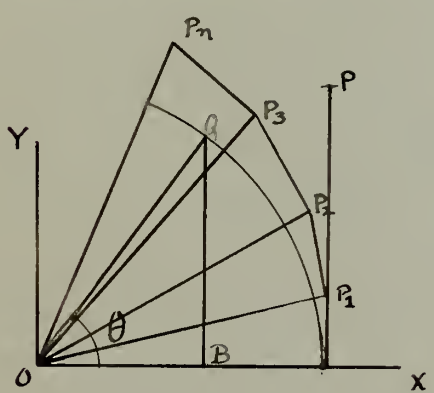


Fig. 2.

Now consider OX the axis of reals and OY the axis of imaginaries, then the directed line OP_1 represents the expression $OA + i\,AP_1 = 1 + \frac{i\theta}{n}$. Construct in succession the similar right triangles OP_1P_2, OP_2P_3, ... $OP_{n-1}P_n$, n in number including

(1) see Moritz, R. E., op. cit. § 170, p. 318-319.
(2) Moritz, R. E., op. cit. pp. 339-340.

the triangle OAP_1, each having for its base the hypotenuse of the triangle immediately preceding, then from the proportionality of the homologous sides we have

$$OP_2 : OP_1 = OP_1 : OA, \text{ or } OP_2 = \overline{OP_1}^2 \text{ (since } OA = 1\text{)},$$

$$OP_3 : OP_2 = OP_2 : OP_1, \text{ or } OP_3 = \overline{OP_1}^3$$

. . . .

Similarly $$OP_n = \overline{OP_1}^n.$$

The directed line OP_1 represents $1 + \frac{i\theta}{n}$,
hence the directed line OP_2 represents $\left(1+\frac{i\theta}{n}\right)^2$,
and the directed line OP_3 represents $\left(1+\frac{i\theta}{n}\right)^3$,
and finally the directed line OP_n represents $\left(1+\frac{i\theta}{n}\right)^n$.

Now let n be indefinitely increased, AP_1 becomes correspondingly smaller. As n increases without limit, the straight lines AP_1, P_1P_2, P_2P_3 ... approach equality, the broken line $AP_1P_2P_3$ ... P_n approaches the arc AQ, and the directed line OP_n approaches the directed line OQ. But the directed line OQ represents $OP + i \cdot PQ = \cos\theta + i \sin\theta$, since $OQ = OA = 1$. Hence, the limit of $\left(1 + \frac{i\theta}{n}\right)^n$, as n increases without limit is $\cos\theta + i \sin\theta$.

4. Exponential Form of the Sine and Cosine.(1) If in

$$e^{i\theta} = \cos\theta + i \sin\theta \qquad (1)$$

we replace θ by $-\theta$, we get

$$e^{-i\theta} = \cos(-\theta) + i \sin(-\theta) = \cos\theta - i \sin\theta, \qquad (2)$$

and solving (1) and (2) for $\cos\theta$ and $\sin\theta$, respectively, we have

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \qquad (3)$$

These relations (3) can be shown to hold for any value of θ, whether real or complex.

(1) Moritz, R. E. op. cit. pp. 340-341.

5. <u>Trigonometry as an Application of Algebra.</u>[1] If we assume a knowledge of imaginaries and exponentials, we might have defined the sine and cosine as in (5) above. With these relations as a basis, the whole of trigonometry becomes an easy application of algebra. All the formulas can be derived from these relations without reference to a geometric triangle.

In this connection it will be well to notice that if in the formula $e^{i\theta} = \cos\theta + i\sin\theta$ we put θ successively equal to $\frac{\pi}{2}$, π and 2π, we obtain the important results:

$$e^{\frac{i\pi}{2}} = i\,, \quad e^{i\pi} = -1\,, \quad e^{2i\pi} = 1\,.$$

Tan θ would be defined as $\dfrac{\sin\theta}{\cos\theta} = \dfrac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}$

$$= \frac{-i(e^{2i\theta} - 1)}{e^{2i\theta} + 1} = -\frac{i(e^{2i\theta} - 1)}{e^{2i\theta} + 1}\,.$$

The reciprocals of $\cos\theta$, $\sin\theta$ and $\tan\theta$ would be called $\sec\theta$, $\csc\theta$ and $\cot\theta$, respectively, that is

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{e^{i\theta} + e^{-i\theta}}{2}} = \frac{2}{e^{i\theta} + e^{-i\theta}} = \frac{2e^{i\theta}}{e^{2i\theta} + 1}\,,$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{e^{i\theta} - e^{-i\theta}}{2i}} = \frac{2i}{e^{i\theta} - e^{-i\theta}} = \frac{2ie^{i\theta}}{e^{2i\theta} - 1}\,,$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{-\frac{i(e^{2i\theta} - 1)}{e^{2i\theta} + 1}} = -\frac{e^{2i\theta} + 1}{i(e^{2i\theta} - 1)} = \frac{i(e^{2i\theta} + 1)}{e^{2i\theta} - 1}\,.$$

The few examples which follow will illustrate how the formulas of trigonometry could be drived from the relations of the previous section.

(1) To prove that $\sin^2 x + \cos^2 x = 1$.

$\sin^2 x + \cos^2 x$

[1] Moritz, R. E., op. cit. pp. 740-741.

$$= \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^2 + \left(\frac{e^{ix} + e^{-ix}}{2}\right)^2$$

$$= \frac{e^{2ix} - 2 + e^{-2ix}}{-4} + \frac{e^{2ix} + 2 + e^{-2ix}}{4}$$

$$= \frac{-e^{2ix} + 2 - e^{-2ix} + e^{2ix} + 2 + e^{-2ix}}{4} = \frac{4}{4} = 1.$$

$\therefore \sin^2 x + \cos^2 x = 1.$

(6) To prove that $\sin 2x = 2 \sin x \cos x$.

$$\sin 2x = \frac{e^{2ix} - e^{-2ix}}{2i} = \frac{(e^{ix} - e^{-ix})(e^{ix} + e^{-ix})}{2i}$$

$$= \frac{2(e^{ix} - e^{-ix})}{2i} \cdot \frac{e^{ix} + e^{-ix}}{2} = 2 \sin x \cos x,$$

$\therefore \sin 2x = 2 \sin x \cos x.$

(7) To prove that $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$,

$$\sin\left(\frac{\pi}{2} - \theta\right) = \frac{e^{\frac{i\pi}{2} - i\theta} - e^{-\frac{i\pi}{2} + i\theta}}{2i}$$

$$= \frac{ie^{-i\theta} - \frac{e^{i\theta}}{i}}{2i} = \frac{-e^{-i\theta} - e^{i\theta}}{-2} = \cos\theta.$$

$\therefore \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta.$

IV Hyperbolic Functions

1. Hyperbolic Functions Defined. Euler's theorem,[1] $\cos\theta + i\sin\theta = e^{i\theta}$ is not limited to real values of θ, but holds true when θ is imaginary or complex, for if we put for $\cos\theta$, $\sin\theta$ and $e^{i\theta}$ (as in section III, 1) the series which define these functions for complex values of θ, the two sides of the equation become identically equal. We may therefore replace θ by $i\theta$ and again by $-i\theta$ and obtain $\cos i\theta + i\sin i\theta = e^{-\theta}$ and $\cos i\theta - i\sin i\theta = e^{\theta}$.

Half the sum and difference give respectively

$$\underline{\cos i\theta = \frac{e^{\theta}+e^{-\theta}}{2}}, \quad \text{and} \quad \underline{-i\sin i\theta = \frac{e^{\theta}-e^{-\theta}}{2}}.$$

For reasons which will appear presently these expressions are known as the hyperbolic cosine of θ and the hyperbolic sine of θ, respectively. The hyperbolic tangent is defined as the ratio of the hyperbolic sine to the hyperbolic cosine. The reciprocals of the hyperbolic cosine, sine and tangent are called the hyperbolic secant, cosecant and cotangent respectively.

Their most common abbreviations are cosh, sinh, tanh, sech, coth. We have accordingly

$$\underline{\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2} = \cos i\theta},$$

$$\underline{\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2} = -i\sin i\theta},$$

(1) Moritz, R. E. op. cit. pp. [illegible].

$$\tanh\theta = \frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}} = -i\tan i\theta,$$

and the reciprocal expressions for sech θ, csch θ and coth θ. Notice that so long as θ is real, e^{θ} and $e^{-\theta}$ are real and each of the hyperbolic functions is real.

3. Duality Between the Circular and Hyperbolic Functions.(1) To every property of, or relation between the trigonometric or circular functions there exists a corresponding property of, or relation between hyperbolic functions. The formulas expressing these properties and relations could all be derived from the expressions of the hyperbolic functions in terms of exponentials, but an easier way to discover them is to substitute the values

$$\cos i\theta = \cosh\theta, \quad \sin i\theta = i\sinh\theta, \quad \tan i\theta = i\tanh\theta, \qquad (1)$$

obtained from the preceding section, in the corresponding formulas for the circular functions.(2)

The following examples will illustrate the method.

1. To find what relation between hyperbolic functions corresponds to the relation $\cos^2\theta + \sin^2\theta = 1$.

Put $i\theta$ for θ, then $\cos^2 i\theta + \sin^2 i\theta = 1$. Substituting for $\cos i\theta$ and $\sin i\theta$ their values from (1) above we have

$\cosh^2\theta + i^2\sinh^2\theta = 1$, or $\cosh^2\theta - \sinh^2\theta = 1$ which is the required relation.

If we check this result by the use of exponentials we obtain

$$\cosh^2\theta = \left(\frac{e^{\theta}+e^{-\theta}}{2}\right)^2 = \frac{e^{2\theta}+2+e^{-2\theta}}{4},$$

$$\sinh^2\theta = \left(\frac{e^{\theta}-e^{-\theta}}{2}\right)^2 = \frac{e^{2\theta}-2+e^{-2\theta}}{4}.$$

(1) Moritz, R. E. op. cit. pp. [illegible].
(2) A good comparative table of formulas for circular and hyperbolic functions can be found in Moritz pp. [illegible]-300.

Subtracting, $\cosh^2\theta - \sinh^2\theta = 1$.

2. To find what relation between hyperbolic functions corresponds to the relation $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

Put $i\theta$ for θ, then $\sin 3i\theta = 3\sin i\theta - 4\sin^3 i\theta$,

or, substituting, $i\sinh 3\theta = 3i\sinh\theta - 4i^3\sinh^3\theta$,

or, $\qquad i\sinh 3\theta = 3i\sinh\theta + 4i\sinh^3\theta$,

or, $\qquad \sinh 3\theta = 3\sinh\theta + 4\sinh^3\theta$.

3. To find the property of hyperbolic functions which corresponds to the property $\sin(\theta \pm 2n\pi) = \sin\theta$.

Put $i\theta$ for θ then

$$\sin(i\theta \pm 2n\pi) = \sin[i(\theta \mp 2ni\pi)] = i\sinh i\theta, \text{ or,}$$

$$\sinh(\theta \mp 2ni\pi) = \sinh\theta .$$

3. Periodicity of Hyperbolic Functions.(1) From problem 3 of the preceding paragraph we see that the function $\sinh\theta$ has a pure imaginary period of $2i\pi$, just as $\sin\theta$ has a real period of 2π. Similarly, it may be shown that each of the hyperbolic functions has the period of $2i\pi$. Also, just as the circular tangent and cotangent have the smaller period of π, so the hyperbolic functions have the period of $i\pi$.

4. Geometrical Representation of Hyperbolic Functions.(2) We will now show that the hyperbolic functions may be expressed as ratios of certain lines connected with the equilateral hyperbola, just as the circular functions are expressed as ratios of certain lines connected with the circle. These relations will be better understood by developing the corresponding results for the circle and hyperbola in parallel columns.

(1) Moritz, R. E., op. cit. p. [illegible].
(2) Moritz, R. E., op. cit. pp. 34[illegible]-350

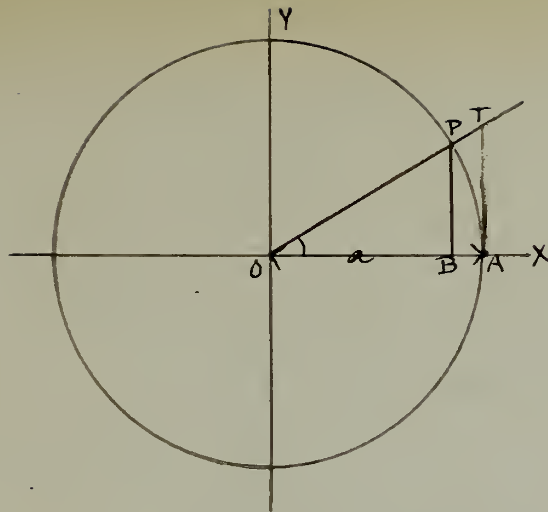


Fig. 3

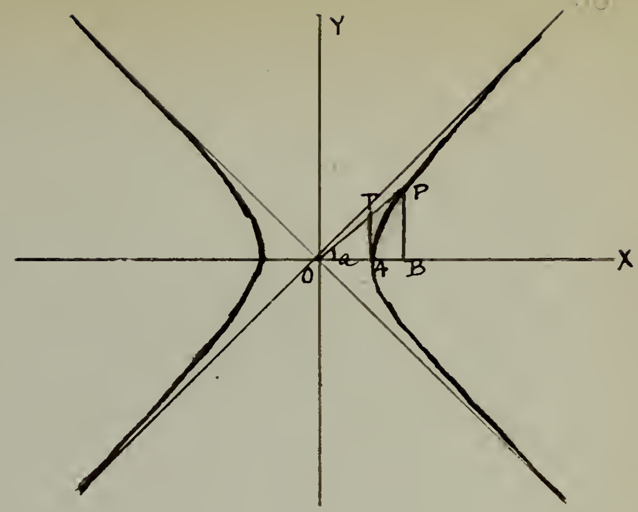


Fig. 4

Let $x = a\cos\theta$, $y = a\sin\theta$,
then since $\cos^2\theta + \sin^2\theta = 1$,
$x^2 + y^2 = a^2(\cos^2\theta + \sin^2\theta) = a^2$,
that is, x and y are the coordinates of a point P on a circle whose radius is a.

$$\cos\theta = \frac{x}{a} = \frac{OB}{OA},$$

$$\sin\theta = \frac{y}{a} = \frac{BP}{OA},$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{BP}{OB} = \frac{AT}{OA}.$$

If OA is taken for the unit of measure,

OB represents $\cos\theta$,
BP represents $\sin\theta$,
AT represents $\tan\theta$.

Let $x = a\cosh u$, $y = a\sinh u$,
then since $\cosh^2 u - \sinh^2 u = 1$,
$x^2 - y^2 = a^2(\cosh^2 u - \sinh^2 u) = a^2$,
that is, x and y are the coordinates of a point P on a hyperbola whose semi-major axis is a.

$$\cosh u = \frac{x}{a} = \frac{OB}{OA},$$

$$\sinh u = \frac{y}{a} = \frac{BP}{OA},$$

$$\tanh u = \frac{\sinh u}{\cosh u} = \frac{BP}{OB} = \frac{AT}{OA}.$$

If OA is taken for the unit of measure,

OB represents $\cosh u$,
BP represents $\sinh u$,
AT represents $\tanh u$.

Thus we see that the hyperbolic functions are related to the hyperbola in the same way that the circular functions are related to the circle. For this reason the functions are called "hyperbolic functions".

I. <u>Inverse Hyperbolic Functions.</u>[1] Let

$$y = \cosh x = \frac{e^x + e^{-x}}{2}, \tag{1}$$

then $x = \cosh^{-1} y$ is called the inverse hyperbolic cosine y.

Put $e^x = z$, then $e^{-x} = \frac{1}{z}$ and we have from (1)

$$z + \frac{1}{z} = 2y, \text{ or } z^2 - 2yz + 1 = 0,$$

from which, on solving for z,

$$e^x = z = y \pm \sqrt{y^2 - 1},$$

and taking the logarithms of both sides of this equation,

$$\underline{x = \cosh^{-1} y = \log\left(y \pm \sqrt{y^2 - 1}\right).} \tag{2}$$

In like manner from $y = \sinh x = \frac{e^x - e^{-x}}{2}$, $x = \sinh^{-1} y$, (3)
the inverse hyperbolic sine y.

Putting in (3) $e^x = z$, we obtain

$$z - \frac{1}{z} = 2y, \text{ or } z^2 - 2yz - 1 = 0.$$

Solving for z, $e^x = z = y \pm \sqrt{y^2 + 1}$.

The minus sign cannot be used, for e^x is positive for every value of x, while $y - \sqrt{y^2 + 1}$ is negative, since $\sqrt{y^2 + 1}$ is greater than y. Hence, on taking the logarithms of both sides of the equation

$$\underline{x = \sinh^{-1} y = \log\left(y + \sqrt{y^2 + 1}\right).} \tag{4}$$

If $y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, then $x = \tanh^{-1} y$, the inverse hyperbolic tangent y.

Again, we put $e^x = z$ and solve the resulting equation for z;

$$e^x = z = \sqrt{\frac{1 + y}{1 - y}},$$

from which

$$\underline{x = \tanh^{-1} y = \frac{1}{2} \log \frac{1 + y}{1 - y}.} \tag{5}$$

(1) Moritz, R. E. op. cit. pp. [illegible]-315.

If we proceed in the same manner we obtain for the remaining inverse hyperbolic functions,

$$x = \coth^{-1} y = \frac{1}{2} \log \frac{y+1}{y-1} \quad , \tag{6}$$

$$x = \operatorname{sech}^{-1} y = \log \frac{1 \pm \sqrt{1-y^2}}{y} \quad , \tag{7}$$

$$x = \operatorname{csch}^{-1} y = \log \frac{1 + \sqrt{1+y^2}}{y} \quad . \tag{8}$$

6. Multiple Values of Inverse Hyperbolic Functions.[1] It follows, from the periodicity of the direct functions (see section IV,3) that the inverse functions, $\sinh^{-1} y$ and $\cosh^{-1} y$, have each an indefinite number of values arranged in a series of intervals of $2i\pi$. Likewise $\tanh^{-1} y$ has an indefinite number of values arranged in a series of intervals of $i\pi$. The particular value of $\sinh^{-1} y$ which lies between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ is called the principal value of $\sinh^{-1} y$. Likewise the particular value of $\cosh^{-1} y$ which lies between 0 and π is called the principal value of $\cosh^{-1} y$, and the particular value of $\tanh^{-1} y$ which lies between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ is called the principal value of $\tanh^{-1} y$.

7. Complex Roots of Cubic Equations.[2] An interesting case of the analogy of hyperbolic to circular functions is found in the expression of the roots of a cubic equation. In the "irreducible case" of the Cardan formula the roots are all real and can be expressed in terms of circular functions. To

(1) McMahon, J., Hyperbolic Functions p. 15.
(2) McMahon, J., op. cit. pp. 45-46.

shall show that analogous expressions in terms of hyperbolic functions can be found for the cases where one root is real and the other two are conjugate complex numbers.

Let us assume that the second degree term of the cubic has been removed and we have it written in the form

$$x^3 \pm 3bx = 2c. \qquad (1)$$

First, let us consider the case where the coefficient of x is -3b and $|c| \leqq |b^{3/2}|$. In this case the roots are all real. Let $x = r \sin\theta$, substitute in (1), and divide by r^3 which gives us

$$\sin^3\theta - \frac{3b}{r^2}\sin\theta = \frac{2c}{r^3}.$$

But $\sin^3\theta - \frac{3}{4}\sin\theta = -\frac{\sin 3\theta}{4}$.(1)

If we let $\frac{3b}{r^2} = \frac{3}{4}$, then $\frac{2c}{r^3} = -\frac{\sin 3\theta}{4}$,

whence $r = \pm 2b^{\frac{1}{2}}$, $\sin 3\theta = \mp\frac{c}{b^{3/2}}$, $\theta = \frac{1}{3}\sin^{-1} \mp \frac{c}{b^{3/2}}$;

therefore, $x = \pm 2b^{\frac{1}{2}}\sin\left(\frac{1}{3}\sin^{-1} \mp \frac{c}{b^{3/2}}\right)$,

or, $$x = -2b^{\frac{1}{2}}\sin\left(\frac{1}{3}\sin^{-1}\frac{c}{b^{3/2}}\right) \quad (2)$$

Now let the principal value of $\sin^{-1}\frac{c}{b^{3/2}}$ as found in the tables, be n; then the three roots will be

$$-2b^{\frac{1}{2}}\sin\frac{n}{3},\ -2b^{\frac{1}{2}}\sin\left(\frac{n}{3} \pm \frac{2\pi}{3}\right), \qquad \text{where } n = \sin^{-1}\frac{c}{b^{3/2}}.$$

(1) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.
(2) $2b^{\frac{1}{2}}\sin\left(\frac{1}{3}\sin^{-1} - \frac{c}{b^{3/2}}\right) = -2b^{\frac{1}{2}}\sin\left(\frac{1}{3}\sin^{-1}\frac{c}{b^{3/2}}\right)$.

Since $-2b^{\frac{1}{2}}\sin\left(\frac{n}{3}\pm\frac{2\pi}{3}\right)$

$$= -2b^{\frac{1}{2}}\left[\sin\frac{n}{3}\cos\frac{2\pi}{3}\pm\cos\frac{n}{3}\sin\frac{2\pi}{3}\right]$$

$$= -2b^{\frac{1}{2}}\left[\left(-\frac{1}{2}\right)\sin\frac{n}{3}\pm\left(\frac{1}{2}\sqrt{3}\right)\cos\frac{n}{3}\right]$$

$$= +\,b^{\frac{1}{2}}\left(\sin\frac{n}{3}\pm\sqrt{3}\cos\frac{n}{3}\right),$$

we can write the three roots as

$$\underline{-2b^{\frac{1}{2}}\sin\frac{n}{3},\quad b^{\frac{1}{2}}\left(\sin\frac{n}{3}\pm\sqrt{3}\cos\frac{n}{3}\right)},\ \text{where } n=\sin^{-1}\frac{c}{b^{3/2}},$$

in which the sign of $b^{\frac{1}{2}}$ is to be taken the same as the sign of c.

Now let us consider the case where the coefficient of x is -3b and $|c|>|b^{3/2}|$. In this case two of the roots are complex numbers. Let $x = r\cosh u$, substitute in (1), and divide by r^3, which gives us

$$\cosh^3 u - \frac{3b}{r^2}\cosh u = \frac{2c}{r^3}.$$

But $\cosh^3 u - \frac{3}{4}\cosh u = \frac{1}{4}\cosh 3u.$(1)

Let $\frac{3b}{r^2}=\frac{3}{4}$, then $\frac{2c}{r^3}=\frac{\cosh 3u}{4}$,

whence $r = \pm 2b^{\frac{1}{2}}$, $\cosh 3u = \pm\frac{c}{b^{3/2}}$, $u=\frac{1}{3}\cosh^{-1}\pm\frac{c}{b^{3/2}}$;

therefore, $x=\pm 2b^{\frac{1}{2}}\cosh\left(\frac{1}{3}\cosh^{-1}\pm\frac{c}{b^{3/2}}\right)$,

or, $x = 2b^{\frac{1}{2}}\cosh\left(\frac{1}{3}\cosh^{-1}\frac{c}{b^{3/2}}\right)$.(2)

Now let the principal value of $\cosh^{-1}\frac{c}{b^{3/2}}$ as found in the tables be n; then two of the imaginary values are $n\pm 2i\pi$ and the three roots are

$$2b^{\frac{1}{2}}\cosh\frac{n}{3},\quad 2b^{\frac{1}{2}}\cosh\left(\frac{n}{3}\pm\frac{2i\pi}{3}\right),\ \text{where } n=\cosh^{-1}\frac{c}{b^{3/2}}.$$

(1) see problem 2, section IV, 2. The development of the formula for $\cosh^3 u$ parallels that which is given for $\sinh^3 u$.

(2) $-2b^{\frac{1}{2}}\cosh\left(\frac{1}{3}\cosh^{-1}-\frac{c}{b^{3/2}}\right) = 2b^{\frac{1}{2}}\cosh\left(\frac{1}{3}\cosh^{-1}\frac{c}{b^{3/2}}\right)$.

Since $2b^{\frac{1}{2}}\cosh\left(\frac{n}{3}\pm\frac{2i\pi}{3}\right)$

$$= 2b^{\frac{1}{2}}\left[\cosh\frac{n}{3}\cosh\frac{2i\pi}{3}\pm\sinh\frac{n}{3}\sinh\frac{2i\pi}{3}\right]$$
$$= 2b^{\frac{1}{2}}\left[\cos\frac{2i^2\pi}{3}\cosh\frac{n}{3}\mp\left(i\sin\frac{2i^2\pi}{3}\right)\sinh\frac{n}{3}\right]$$
$$= 2b^{\frac{1}{2}}\left[\left(-\frac{1}{2}\right)\cosh\frac{n}{3}\pm\left(\frac{i}{2}\sqrt{3}\right)\sinh\frac{n}{3}\right]$$
$$= b^{\frac{1}{2}}\left(\cosh\frac{n}{3}\pm i\sqrt{3}\sinh\frac{n}{3}\right),$$

we can write the three roots as,

$\underline{2b^{\frac{1}{2}}\cosh\frac{n}{3},\ -b^{\frac{1}{2}}\left(\cosh\frac{n}{3}\pm i\sqrt{3}\sinh\frac{n}{3}\right)}$, where $n=\cosh^{-1}\frac{c}{b^{3/2}}$,

in which the sign of $b^{\frac{1}{2}}$ is to be taken the same as the sign of c.

Let us finally consider the case where b has the positive sign. In this case two of the roots are complex numbers. Let $x = r\sinh u$, substitute, and divide by r^3, then

$$\sinh^3 u+\frac{3b}{r^2}\sinh u=\frac{2c}{r^3}.$$

But $\sinh^3 u+\frac{3}{4}\sinh u=\frac{1}{4}\sinh 3u.$(1)

Let $\frac{3b}{r^2}=\frac{3}{4}$, then $\frac{2c}{r^3}=\frac{\sinh 3u}{4}$,

whence $r=\pm 2b^{\frac{1}{2}}$, $\sinh 3u=\frac{c}{b^{3/2}}$, $u=\frac{1}{3}\sinh^{-1}\pm\frac{c}{b^{3/2}}$;

therefore $x=\pm 2b^{\frac{1}{2}}\sinh\left(\frac{1}{3}\sinh^{-1}\pm\frac{c}{b^{3/2}}\right)$,

or $x=2b^{\frac{1}{2}}\sinh\left(\frac{1}{3}\sinh^{-1}\frac{c}{b^{3/2}}\right)$.(2)

Now let the principal value of $\sinh^{-1}\frac{c}{b^{3/2}}$ as found in the tables be n; then two of the imaginary values are $n\pm 2i\pi$, and the three roots are

$2b^{\frac{1}{2}}\sinh\frac{n}{3}$, $2b^{\frac{1}{2}}\sinh\left(\frac{n}{3}\pm\frac{2i\pi}{3}\right)$, where $n=\sinh^{-1}\frac{c}{b^{3/2}}$.

(1) see problem 2, section IV, 9.
(2) $-2b^{\frac{1}{2}}\sinh\left(\frac{1}{3}\sinh^{-1}-\frac{c}{b^{3/2}}\right)=2b^{\frac{1}{2}}\sinh\left(\frac{1}{3}\sinh^{-1}\frac{c}{b^{3/2}}\right)$.

Since $2b^{\frac{1}{2}} \sinh\left(\frac{n}{3} \pm \frac{2i\pi}{3}\right)$

$$= 2b^{\frac{1}{2}}\left[\sinh\frac{n}{3}\cosh\frac{2i\pi}{3} \pm \cosh\frac{n}{3}\sinh\frac{2i\pi}{3}\right]$$

$$= 2b^{\frac{1}{2}}\left[\cos\frac{2i^2\pi}{3}\sinh\frac{n}{3} \mp \left(i\sin\frac{2i^2\pi}{3}\right)\cosh\frac{n}{3}\right]$$

$$= 2b^{\frac{1}{2}}\left[\left(-\frac{1}{2}\right)\sinh\frac{n}{3} \pm \left(\frac{i}{2}\sqrt{3}\right)\cosh\frac{n}{3}\right]$$

$$= -b^{\frac{1}{2}}\left(\sinh\frac{n}{3} \pm i\sqrt{3}\cosh\frac{n}{3}\right),$$

we can write the three roots as,

$2b^{\frac{1}{2}}\cosh\frac{n}{3}$, $-b^{\frac{1}{2}}\left(\sinh\frac{n}{3} \pm i\sqrt{3}\cosh\frac{n}{3}\right)$, where $n = \sinh^{-1}\frac{c}{b^{3/2}}$,

in which the sign of $b^{\frac{1}{2}}$ is to be taken the same as the sign of c.

The three examples which follow will illustrate each of the cases in turn.

1. $x^3 - 12x - 10 = 0$

$$x^3 - 3(4)x = 2(5)$$

$$x = -2b^{\frac{1}{2}}\sin\frac{n}{3}$$

$$= -2(2)\sin 12^\circ 53' 40''$$

$$= -4(.2232) = \underline{-.8928}$$

$$n = \sin^{-1}\frac{c}{b^{3/2}} = \sin^{-1}\frac{5}{8}$$

$$= \sin^{-1}.6250 = 38^\circ 41'.$$

$$\frac{n}{3} = 12^\circ 53' 40''.$$

$$x = b^{\frac{1}{2}}\left(\sin\frac{n}{3} \pm \sqrt{3}\cos\frac{n}{3}\right)$$

$$= 2(.2232 \pm 1.7321 \cdot .9748)$$

$$= 2(.2232 \pm 1.6884)$$

$$= 2(1.9116) \text{ and } 2(-1.4652)$$

$$= \underline{3.8232}, \text{ and } \underline{-2.9304}.$$

2. $27x^3-36x+65=0$

$$x^3-3\left(\tfrac{4}{9}\right)x=2\left(-\tfrac{65}{54}\right)$$

$$x=2b^{\frac{1}{2}}\cosh\frac{n}{3}$$

$$=2\left(-\tfrac{2}{3}\right)\cosh .6932$$

$$=-\tfrac{4}{3}(1.250)=\underline{-1.667.}$$

$$n=\cosh^{-1}\frac{c}{b^{3/2}}=\cosh^{-1}\frac{-\frac{65}{54}}{-\frac{8}{27}}$$

$$=\cosh^{-1}\frac{65}{16}=\cosh^{-1}4.0625$$

$$=2.0795.$$

$$\frac{n}{3}=.6932.$$

$$x=-b^{\frac{1}{2}}\left(\cosh\frac{n}{3}\pm i\sqrt{3}\sinh\frac{n}{3}\right)$$

$$=\frac{2}{3}\left[1.250\pm1.732i(.7501)\right]$$

$$=\underline{.833\pm.866i.}$$

3. $x^3+9x-26=0$

$$x^3+3(3)x=2(13)$$

$$x=2b^{\frac{1}{2}}\sinh\frac{n}{3}$$

$$=2\sqrt{3}\sinh .5493$$

$$=2\cdot1.7321\cdot.5773$$

$$=\underline{2.000.}$$

$$n=\sinh^{-1}\frac{c}{b^{3/2}}=\sinh^{-1}\frac{13}{\sqrt{27}}$$

$$=\sinh^{-1}\frac{13\sqrt{3}}{9}=\sinh^{-1}\frac{13}{9}\cdot1.7321$$

$$=\sinh^{-1}2.5019=1.6479.$$

$$\frac{n}{3}=.5493.$$

$$x=-b^{\frac{1}{2}}\left(\sinh\frac{n}{3}\pm i\sqrt{3}\cosh\frac{n}{3}\right)$$

$$=-\sqrt{3}\left(.5773\pm i\sqrt{3}(1.1547)\right)$$

$$=-1.7321(.5773)\pm i(1.1547)$$

$$=\underline{-1.000\pm3.464i.}$$

8. Use of Hyperbolic Functions.(1) Numerous uses are made of hyperbolic functions in applied mathematics. While it is beyond the scope of this paper to go into these applications, we will indicate to some extent their usefulness. C. D. Walcott, secretary of the Smithsonian Institution makes the following statement.

"Hyperbolic functions are extremely useful in every branch of physics and in the application of physics, whether to observational and experimental sciences or to technology. Thus, whenever, an entity (such as light, velocity, electricity, or radio activity) is subject to gradual extinction or absorption, the decay is represented by some form of hyperbolic functions. Mercator's projection is likewise computed by hyperbolic functions. Whenever mechanical strains are regarded as great enough to be measured they are most simply expressed in terms of hyperbolic functions. Hence geological deformations invariably lead to such expressions." (2)

(1) Moritz, R. E., op. cit. p. [illegible].
(2) Mahon, J. op. cit. p. 31, p. [illegible].
Applications of hyperbolic functions will be found in the following volumes. Rayleigh, Theory of Sound, Vol. I; Love, Elasticity; Byerly, Fourier Series; Bassett, Hydrodynamics; Maxwell, Electricity; Gunther, Hyperbelfunktionen, and Kennelly, Application of Hyperbolic Functions to Electrical Engineering Problems.
The tables prepared by Becker and Van Orstrand are published by the Smithsonian Institution, and those of Professor A. E. Kennelly are published by the Harvard University Press.

V Use of Complex Numbers in the Solution of Several Isolated Problems

1. Proof of Euler's Theorem, $(a^2+b^2+c^2+d^2)(x^2+y^2+z^2+w^2)=m^2+n^2+p^2+q^2$

If we write $a^2+b^2+c^2+d^2$ as the determinant,

$$\begin{vmatrix} a+bi & -c+di \\ c+di & a-bi \end{vmatrix}$$

and $x^2+y^2+z^2+w^2$ as the determinant,

$$\begin{vmatrix} x+yi & z+wi \\ -z+wi & x-yi \end{vmatrix}$$

then the product $(a^2+b^2+c^2+d^2)(x^2+y^2+z^2+w^2)$

$$= \begin{vmatrix} a+bi & -c+di \\ c+di & a-bi \end{vmatrix} \cdot \begin{vmatrix} x+yi & z+wi \\ -z+wi & x-yi \end{vmatrix}$$

$$= \begin{vmatrix} (a+bi)(x+yi)+(-c+di)(-z+wi) & (a+bi)(z+wi)+(-c+di)(x-yi) \\ (c+di)(x+yi)+(a-bi)(-z+wi) & (c+di)(z+wi)+(a-bi)(x-yi) \end{vmatrix} \quad (1)$$

$$= \begin{vmatrix} (ax-by+cz-dw)+(bx+ay-dz-cw)i & (az-bw-cx+dy)+(bz+aw+dx+cy)i \\ (cx-dy-az+bw)+(dx+cy+bz+aw)i & (cz-dw+ax-by)+(dz+cw-bx-ay)i \end{vmatrix}$$

Let $m=ax-by+cz-dw$, $n=bx+ay-dz-cw$,

$p=cx-dy-az+bw$, $q=dx+cy+bz+aw$,

then $\begin{vmatrix} m+ni & -p+qi \\ p+qi & m-ni \end{vmatrix} = m^2+n^2+p^2+q^2.$

Therefore, a product of two factors each of which is the sum of four squares is itself the sum of four squares.

(1) The product of two determinants of the nth order may be expressed as a determinant of the nth order in which the element which lies in the rth row, and the kth column is obtained by multiplying each element of the rth row of the first factor by the corresponding element of the kth column in the second factor and adding the results.

8. <u>Decomposition of a Rational Fraction into Partial Fractions when x^2+1 is a Factor of the Denominator.</u> Let

$\dfrac{X^m}{X_1^n}$, where $m < n$, be equal to $\dfrac{F(x)}{(x-a)(x-b)^2(x^2+1)}$,

Then $\dfrac{F(x)}{(x-a)(x-b)^2(x^2+1)}$ can be decomposed into the partial fractions $\dfrac{Ax+B}{(x^2+1)} + \dfrac{Q}{(x-a)(x-b)^2}$,

$$\text{or } \frac{F(x)}{(x-a)(x-b)^2(x^2+1)} = \frac{Ax+B}{x^2+1} + \frac{Q}{(x-a)(x-b)^2}$$

$$\text{or } F(x) = (Ax+B)(x-a)(x-b)^2 + Q(x^2+1). \qquad (1)$$

If we let $x = i$, then $x^2+1 = 0$, and substituting in (1) we have $F(i) = (Ai+B)(i-a)(i-b)^2$,

or $\alpha + \beta i = (\gamma A + \delta B)i + \epsilon A + \zeta B$,

or $(\beta - \gamma A - \delta B)i = \epsilon A + \zeta B - \alpha$, an equality which is impossible unless each member equals zero. Hence

$$\left.\begin{aligned} \beta - \gamma A - \delta B &= 0 \\ \epsilon A + \zeta B - \alpha &= 0 \end{aligned}\right\}$$

from which simultaneous equations the unknown A and B can be determined.

The partial fractions corresponding to $\dfrac{Q}{(x-a)(x-b)^2}$ are determined without the use of imaginaries. (1)

For example (2), to determine the partial fraction, corresponding to x^2+1 in the decomposition of

$$\frac{4x^4-16x^3+17x^2-8x+7}{(x-1)(x-2)^2(x^2+1)} = \frac{Ax+B}{x^2+1} + \frac{Q}{(x-1)(x-2)^2}.$$

(1) see Chrystal, G., Algebra, Part I, pp. 156-157.
(2) from Chrystal, G. op. cit. p. 157-158.

Multiplying both sides of the equation by the lowest common denominator, we have

$$4x^4 - 13x^3 + 17x^2 - 8x + 7 = (Ax+B)(x-1)(x-2)^2 + C(x^2+1).$$

Let $x = i$, then we have

$$8i - 6 = -7A + B + 7iB + iA$$

whence $(7B + A - 8)i = -B + 7A - 6.$

Since both members must equal 0, then

$$7B + A - 8 = 0$$

$$-B + 7A - 6 = 0,$$

and $A = 1$, $B = 1$.

Therefore, $\dfrac{Ax+B}{x^2+1} = \dfrac{x+1}{x^2+1}$.

3. <u>Solution of $\int e^{ax} \sin bx\, dx$</u>. If we assume that the laws established for the integration of real numbers are true for complex numbers, then by using the exponential forms of the sine and cosine (section III, 4) the $\int e^{ax} \sin bx\, dx$ can be found without much difficulty.

$$\int e^{ax} \sin bx\, dx = \int e^{ax}\left(\frac{e^{ibx} - e^{-ibx}}{2i}\right) dx$$

$$= \frac{1}{2i}\int e^{ax+ibx} dx - \frac{1}{2i}\int e^{ax-ibx} dx$$

$$= \frac{1}{2i(a+bi)}\int e^{ax+ibx}(a+bi)\, dx - \frac{1}{2i(a-bi)}\int e^{ax-ibx}(a-bi)\, dx$$

$$= \frac{1}{2i(a+bi)}\left(e^{ax+ibx}\right) - \frac{1}{2i(a-bi)}\left(e^{ax-ibx}\right) + C$$

$$= \frac{ae^{ax+ibx} - bie^{ax+ibx} - ae^{ax-ibx} - bie^{ax-ibx}}{2i(a^2+b^2)} + C$$

$$= \frac{ae^{ax}(e^{ibx} - e^{-ibx}) - ibe^{ax}(e^{ibx} + e^{-ibx})}{2i(a^2+b^2)} + C$$

$$= \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2+b^2} + C$$

4. Solution of $\int e^{ax}\cos bx\,dx$. In a similar manner the $\int e^{ax}\cos bx\,dx$ can be found.

$$\int e^{ax}\cos bx\,dx = \int e^{ax}\left(\frac{e^{ibx}+e^{-ibx}}{2}\right)dx$$

$$= \frac{1}{2}\int e^{ax+ibx}\,dx + \frac{1}{2}\int e^{ax-ibx}\,dx$$

$$= \frac{1}{2(a+bi)}\int e^{ax+ibx}(a+bi)\,dx + \frac{1}{2(a+bi)}\int e^{ax+ibx}(a-bi)\,dx$$

$$= \frac{1}{2(a+bi)}\left(e^{ax+ibx}\right) + \frac{1}{2(a-bi)}\left(e^{ax-ibx}\right) + C$$

$$= \frac{e^{ax+ibx}}{2(a+bi)} + \frac{e^{ax-ibx}}{2(a-bi)} + C$$

$$= \frac{ae^{ax+ibx} - ibe^{ax+ibx} + ae^{ax-ibx} + ibe^{ax-ibx}}{2(a^2+b^2)} + C$$

$$= \frac{ae^{ax}(e^{ibx}+e^{-ibx}) - ibe^{ax}(e^{ibx}-e^{-ibx})}{2(a^2+b^2)} + C$$

$$= \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} + C$$

5. Solution of Linear Differential Equations with Constant Coefficients. (Roots of Auxiliary Equation Complex.)(1)

The general linear differential equation with constant coefficients is

$$k_0\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + k_{n-1}\frac{dy}{dx} + k_ny = X, \qquad (1)$$

where $k_0, k_1, \ldots\ldots, k_n$ are constants.

If we let $\frac{dy}{dx} = Dy$, $\frac{d^2y}{dx^2} = D^2y$, $\cdots$ $\frac{d^ny}{dx^n} = D^ny$, we can write (1) in the following form.

$$(k_0D^n + k_1D^{n-1} + k_2D^{n-2} + \cdots + k_{n-1}D + k_n)y = X, \qquad (2)$$

$$\text{or} \quad f(D)y = X. \qquad (3)$$

(1) Cohen: An Elementary Treatise of Differential Equations, pp. [illegible].

If $X = 0$, then we have a homogeneous linear differential equation, the general integral of which is $c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$, where $y_1, y_2, \ldots, y_n$ are n linearly independent particular integrals of this equation.(1)

If $X \neq 0$, then we have a complete linear differential equation. For its solution we first make X temporarily equal to zero. The solution of this equation is called the complementary function of the complete linear differential equation. The general integral of a complete linear differential equation is the sum of its complementary function and any particular integral.

In the equation (3) suppose $X = 0$. Then

$$f(D)\,y = 0. \tag{4}$$

Letting $y = e^{mx}$, we have $Dy = me^{mx}, \ldots\ldots, D^r y = m^r e^{mx}$,

hence, $f(D)(e^{mx}) = e^{mx} f(m)$.

For e^{mx} to be an integral of (4), m must satisfy the equation

$$f(m) = 0, \tag{5}$$

i.e. $k_0 m^n + k_1 m^{n-1} + k_2 m^{n-2} + \cdots + k_{n-1} m + k_n = 0.$

Each value of m satisfying (5) gives an integral of (4).

If these are all distinct, (say $m_1, m_2, m_3, \ldots, m_n$), $e^{m_1 x}$, $e^{m_2 x}, \ldots\ldots, e^{m_n x}$ will be linearly independent, and $c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots\cdots c_n e^{m_n x}$ will be the general integral of (4), and the complementary function of (3).

Equation (5) we shall call the auxiliary equation. If any roots of the auxiliary equation are repeated the above

(1) See Cohen pp. [illegible].

method does not give us n linearly independent integrals, and consequently it does not give us the general solution. In this case we make the more general substitution $y = e^{mx}\phi(x)$, where $\phi(x)$ is a function of x entirely at our disposal. Then

$$Dy = e^{mx}[m\phi + D\phi],$$
$$D^2y = e^{mx}[m^2\phi + 2mD\phi + D^2\phi],$$
$$D^3y = e^{mx}[m^3\phi + 3m^2D\phi + 3mD^2\phi + D^3\phi],$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$D^ny = e^{mx}\Big[m^n\phi + n\,m^{n-1}D\phi + \frac{n(n-1)}{2!}m^{n-2}D^2\phi + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r!}m^{n-r}D^r\phi + \cdots + D^n\phi\Big],$$

whence
$$f(D)y = e^{mx}\Big[f(m)\phi + f'(m)D\phi + \frac{1}{2!}f''(m)D^2\phi + \cdots + \frac{1}{r!}f^{(r)}(m)D^r\phi + \cdots + \frac{1}{n!}f^{(n)}(m)D^n\phi\Big],$$

where $f'(m) = \frac{d}{dm}f(m), \cdots, f^{(r)}(m) = \frac{d^r}{dm^r}f(m).$

If m_1 is an r-fold root of $f(m)=0$, then

$$f(m_1) = 0,\ f'(m_1) = 0,\ \cdots,\ f^{(r-1)}(m_1) = 0.\text{(1)}$$

In this case $f(D)y$ will vanish if $y = e^{mx}\phi(x)$ provided $D^r\phi = 0$, whence all the higher derivatives of ϕ are also zero; i.e. provided $\phi = c_1x^{r-1} + c_2x^{r-2} + \cdots + c_{r-1}x + c_r$, where $c_1, c_2, \ldots, c_r$ are any constants whatever. Hence, we see that if m_1 corresponds to an r-fold root of the auxiliary equation, not only is e^{m_1x} an integral of the equation, but so also are $xe^{m_1x}, x^2e^{m_1x}, \ldots, x^{r-1}e^{m_1x}$, i.e. corresponding to an r-fold root we have r linearly independent integrals. So that whether the roots of the auxiliary

(1) see Cohen, op. cit. p. 65.

equation are repeated or not, the n linearly independent integrals necessary for obtaining the complementary function are always supplied by the auxiliary equation.

If the coefficients of the differential equation are real, while some or all of the roots of the auxiliary equation are not, we can, by a proper arrangement of the terms in the complementary function, have the latter involve only real terms. Thus, if the auxiliary equation has a root $\alpha+i\beta$, it also has $\alpha-i\beta$ as a root, since its coefficients are real. Two terms of the complementary function will then be

$$c_1 e^{(\alpha+i\beta)x}+c_2 e^{(\alpha-i\beta)x},$$

or

$$e^{\alpha x}(c_1 e^{i\beta x}+c_2 e^{-i\beta x}).$$

Now $e^{i\beta x}=\cos\beta x+i\sin\beta x$, and $e^{-i\beta x}=\cos\beta x-i\sin\beta x$.

Hence one pair of terms may be written

$$e^{\alpha x}\left[(c_1+c_2)\cos\beta x+i(c_1-c_2)\sin\beta x\right].$$

Putting $c_1=\frac{A-iB}{2}$, $c_2=\frac{A+iB}{2}$ we have

$$\underline{e^{\alpha x}(A\cos\beta x+B\sin\beta x),}$$

where A and B are the two arbitrary constants.

For interpreting the solution in physical problems, another form is sometimes preferable.

$A\cos\beta x+B\sin\beta x$ may be written $\sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\cos\beta x+\frac{B}{\sqrt{A^2+B^2}}\sin\beta x\right)$.

Since the sum of the squares of $\frac{A}{\sqrt{A^2+B^2}}$ and $\frac{B}{\sqrt{A^2+B^2}}$ equals unity, these may be taken as the sine and cosine of some angle, say h.

Putting $\sqrt{A^2+B^2} = a$ our expression becomes either

$$a(\sin b \cos\beta x + \cos b \sin\beta x), \text{ or } a(\cos b \cos\beta x + \sin b \sin\beta x)$$

which equals either $a\sin(\beta x + b)$ or $a\cos(\beta x - b)$ respectively. Hence, $e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$$= a e^{\alpha x}\sin(\beta x + b) = a e^{\alpha x}\cos(\beta x - b).$$

It is obvious, that in case a pair of such roots is repeated, the corresponding part of the complementary function is

$$e^{\alpha x}(A_1\cos\beta x + B_1\sin\beta x) + x e^{\alpha x}(A_2\cos\beta x + B_2\sin\beta x)$$

$$\text{or } e^{\alpha x}\left[(A_1 + A_2 x)\cos\beta x + (B_1 + B_2 x)\sin\beta x\right],$$

and perfectly generally, in case such a pair occur as r-fold roots, the corresponding part of the complementary function is

$$e^{\alpha x}\left[(A_1 + A_2 x + \cdots + A_r x^{r-1})\cos\beta x + (B_1 + B_2 x + \cdots + B_r x^{r-1})\sin\beta x\right].$$

The following problem will illustrate the principle.

$$(D^4 + 2D^2 + 1)y = 0.$$

Let $y = e^{mx}$, then $(D^4 + 2D^2 + 1)e^{mx} = e^{mx}f(m)$.

If $m^4 + 2m^2 + 1 = 0$,

then $m = \pm\sqrt{-1}, \pm\sqrt{-1}$,

$\alpha = 0, \ \beta = 1,$

$$\text{Hence } y = e^{\alpha x}\left[(A_1 + A_2 x)\cos\beta x + (B_1 + B_2 x)\sin\beta x\right],$$

$$= \left[(A_1 + A_2 x)\cos x + (B_1 + B_2 x)\sin x\right].$$

7. Solution of Homogeneous Linear Partial Differential Equations with Constant Coefficients (Roots of Auxiliary Complex)[1] There are many points of similarity between the linear partial differential equation of the nth order and the linear ordinary differential equation of the nth order. If we put $D = \dfrac{\partial}{\partial x}$, $\mathcal{D} = \dfrac{\partial}{\partial y}$, the general type of this equation is

$$(P_{n,0}D^n + P_{n-1,1}D^{n-1}\mathcal{D} + P_{n-2,2}D^{n-2}\mathcal{D}^2 + \cdots + P_{0,n}\mathcal{D}^n + P_{n-1,0}D^{n-1} +$$

(1) Cohen, A, op. cit. pp. 238-240, 242, 243.

$$\cdots + P_{3,r} D^s \mathcal{D}^r + \cdots + P_{1,0} D + P_{0,1} \mathcal{D} + P_{0,0}) z = f(x, y),$$

or more briefly

$$F(D, \mathcal{D}) z = f(x, y), \tag{1}$$

where $F(D, \mathcal{D})$ is a symbolic operator, which, looked upon algebraically, is a polynomial of degree n in D and $\mathcal{D}$.

Obviously, $F(D, \mathcal{D})(u+v) = F(D, \mathcal{D}) u + F(D, \mathcal{D}) v$. Hence the problem of solving (1) can be divided into two, viz. that of finding the general integral of

$$F(D, \mathcal{D}) z = 0, \tag{2}$$

which we shall call the <u>complementary function</u> of (1), and that of finding any particular integral. The sum of these two will give the general integral of (1).

As in our discussion of the ordinary linear differential equation, we shall use the term, <u>homogeneous</u>, to apply to an equation in which all the derivatives are of the same order. In this case the symbolic operator is homogeneous in D and $\mathcal{D}$. Let us suppose also that the coefficients are constants, and the right member is zero. Our equation will be of the form

$$(k_0 D^n + k_1 D^{n-1}\mathcal{D} + \cdots k_{n-1} D \mathcal{D}^{n-1} + k_n \mathcal{D}^n) z = 0 \tag{3}$$

or $F(D, \mathcal{D}) z = 0$.

Since for ϕ, any function whatever,

$$D^r \mathcal{D}^s \phi(y + mx) = m^r \phi^{(r+s)}(y + mx),$$

where $\phi^{r+s}(y+mx)$ means $\dfrac{d^{r+s}\phi(y+mx)}{[d(y+mx)]^{r+s}}$, the result

of substituting $z=\phi(y+mx)$ in (3) will be

$$\phi^{(n)}(y+mx)\,F(m,1)=0$$

Hence, $z=\phi(y+mx)$ will be a solution, provided $F(m,1)=0$;

i.e. $k_0 m^n + k_1 m^{n-1} + \cdots k_{n-1} m + k_n = 0.$ (4)

If the roots of (4), which we shall call the auxiliary equation, are distinct, say $m_1, m_2, \ldots\ldots, m_n$,

$$z=\phi_1(y+m_1x)+\phi_2(y+m_2x)+\cdots+\phi_n(y+m_nx)$$

will be a solution. Since it contains n arbitrary functions, it will be the general solution.

Let us assume that some of the roots of this auxiliary equation are complex. If the coefficients in the differential equation are real, the complex roots of the auxiliary equation occur in pairs of conjugates. Hence, if $\alpha+i\beta$ is a root, $\alpha-i\beta$ will also be one. The corresponding terms in the complementary function will be

$$\phi(y+\alpha x+i\beta x)+\psi(y+\alpha x-i\beta x).$$

If ϕ_1 and ψ_1 are any two arbitrarily chosen functions, there is no loss in putting

$$\phi=\phi_1+i\psi_1,\quad \psi=\phi_1-i\psi_1.$$

The expression above then becomes

$$\phi_1(y+\alpha x+i\beta x)+\phi_1(y+\alpha x-i\beta x)+i\left[\psi_1(y+\alpha x+i\beta x)-\psi_1(y+\alpha x-i\beta x)\right].$$

For ϕ_1 and ψ_1, any real functions this expression is real.

For example, if $\dfrac{\partial^2 z}{\partial x^2}-2\dfrac{\partial^2 z}{\partial x\partial y}+2\dfrac{\partial^2 z}{\partial y^2}=0$, the auxiliary equation is $m^2-2m+2=0$. Therefore $m=1\pm i$.

The general solution is

$$z = \phi(y + x + ix) + \psi(y + x - ix).$$

Letting $\phi = \phi_1 + i\psi_1$ and $\psi = \phi_1 - i\psi_1$, we give it the real form

$$z = \phi_1(y+x+ix) + \phi_1(y+x-ix) + i[\psi_1(y+x+ix) - \psi_1(y+x-ix)],$$

for ϕ_1 and ψ_1, any real functions.

If, in particular we choose $\phi_1(u)$ to be cos u, and $\psi_1(u)$ to be e^u we have

$$\cos(y+x+ix) = \cos(x+y)\cos ix - \sin(x+y)\sin ix$$
$$= \cos(x+y)\cosh x - i\sin(x+y)\sinh x,$$
$$\cos(y+x-ix) = \cos(x+y)\cos ix + \sin(x+y)\sin ix$$
$$= \cos(x+y)\cosh x + i\sin(x+y)\sinh x,$$
$$e^{y+x+ix} - e^{y+x-ix} = e^{y+x}(e^{ix} - e^{-ix}) = 2ie^{y+x}\sin x.$$

Therefore, $z = 2\cos(x+y)\cosh x - 2e^{x+y}\sin x$.

VI Conformal Representation and Map Drawing

1. <u>The Function $\frac{1}{z}$ and the Transformation by Reciprocal Radii.</u>[1] Let us consider what transformation of the plane into itself is determined by the function

$$z' = \frac{1}{z} \tag{1}$$

To investigate this, put

$$z = x + iy = r(\cos\phi + i\sin\phi),$$
$$z' = x' + iy' = r'(\cos\phi' + i\sin\phi');$$

from which we obtain

$$r' = \frac{1}{r}, \quad \phi' = -\phi, \tag{2}$$

and therefore

$$x' = \frac{x}{x^2+y^2}, \qquad y' = \frac{-y}{x^2+y^2}. \tag{3}$$

The transformation determined by these equations may be regarded as compounded from two simpler geometric transformations, each of which considered by itself is not determined by rational functions of a complex variable. Let us introduce the auxiliary transformation:

$$\overline{r} = r, \qquad \overline{\phi} = -\phi, \tag{4}$$

or

$$\overline{x} = x, \qquad \overline{y} = -y, \tag{5}$$

and therefore to obtain transformation (1) we must after this put

$$r' = \frac{1}{\overline{r}}, \qquad \phi' = \overline{\phi}, \tag{6}$$

$$x' = \frac{\overline{x}}{\overline{x}^2+\overline{y}^2}, \quad y' = \frac{\overline{y}}{\overline{x}^2+\overline{y}^2}. \tag{7}$$

(1) Burkhardt-Rasor, Theory of Functions of a Complex Variable, p. [illegible].

4. Equations ([illegible]), ([illegible]) effect the transition from any complex number to its conjugate, thus determining geometrically a reflection on the axis of reals.

5. The transformation determined by equations ([illegible]) is called (on account of the first one of them) the transformation by reciprocal radii with reference to the unit circle--also called reflection on the unit circle.

The transformation by reciprocal radii has the property that angles are preserved, that is, that the angle of intersection of two curves is equal to the angle of intersection of the corresponding curves. The correctness of the statement can be shown best by considering first the special case in which one of the two curves is a straight line through the origin.

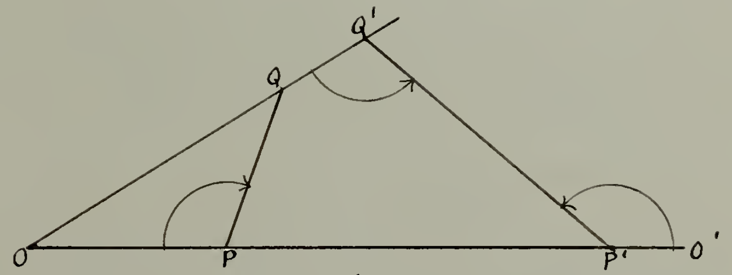


Fig. 5.

Thus if PP' and QQ' are two pairs of corresponding points (Fig. 5), it then follows according to equation ([illegible]) that

$$\overline{OP}\cdot\overline{OP'} = \overline{OQ}\cdot\overline{OQ'} = 1;$$

and hence $\Delta\, OPQ \sim \Delta\, OQ'P'$

and in particular

$$\angle OPQ = \angle OQ'P'. \tag{8}$$

If we allow the point Q to approach the point P along a given curve, then the point Q' approaches the point P' upon the corresponding curve; PQ, $P'Q'$ become the directions of the tangents to the curves, $\angle OQ'P'$ in the limit will be equal to $\angle OP'Q'$, and it therefore follows that in the limit

$$\angle OPQ = \angle OP'Q'. \tag{9}$$

We notice further in this connection that the equality sign refers only to the absolute value of the angle, the two angles corresponding to each other are opposite in sense, and the resulting theorem is completely formulated as follows:

Two curves corresponding to each other form with any straight line corresponding to itself, angles which are equal but of opposite sense.

Moreover, since the angle between any two lines is equal to the sum (difference, respectively) of the two angles which the two lines make with a third, it follows that:

C. <u>The angle in which any two curves intersect is equal in the opposite sense to the angle of intersection of the two curves which correspond to the first two by the transformation by reciprocal radii.</u>

Since this transformation is of frequent occurrence, a name is given to it as in the following definition.

D. <u>A transformation under which the angle between any two curves is equal to the angle between the corresponding curves, is called a conformal representation (also isogonal represent-</u>

ation, or a mapping with preservation of angles, or the 'similar in infinitesimal parts'.)(1)

Therefore, according as the sense of the angle obtained is preserved or changed we speak of the representation as conformal "without" or "with inversion of angles". With this terminology Theorem C is stated as follows:

E. The transformation by reciprocal radii is a conformal representation with inversion of angles.

Reflection on the x-axis determined by equations ([illegible]) or (3) is a representation of the same kind. If we now combine transformations ([illegible]) and ([illegible]) in order to obtain the original transformation (1), the two changes in the sense of the angle, being opposites, mutually disappear. We can then say and it is the most important result of this discussion:

F. The transformation effected by $z = \frac{1}{z}$ is a conformal representation without inversion of angles.

(1) "The characteristic property of the copy is that angles are preserved, and that no change is made in the relative positions, and (save as to uniform magnification) no change is made in the relative distances of points that lie in the immediate vicinity of a given point. The leading feature of this property is maintained over the whole copy for every small element of area; but the magnification which is uniform for each element is not uniform over the whole of the copy." A. R. Forsyth, Sc. D. F.R.S., Theory of Functions of a Complex Variable, p. 531. For a discussion of conformal representation from the standpoint of analytic functions see Burkhardt-Rasor, Theory of Functions of a Complex Variable, pp. [illegible], or Curtiss, Analytic Functions of a Complex Variable, pp. [illegible].

2. Transition from the Plane to the Sphere by Stereographic Projection.(1) Complex numbers may be represented by the points on a spherical surface as well as by points in a plane. We proceed as follows:

A. Place a sphere of unit diameter on the xy-plane (considered horizontal) so that it touches the plane at the origin O. The highest point of the sphere—that one which lies diametrically opposite to O—will be called O'. From this point O', project the points of the plane on the sphere by straight lines. (See Fig. 6).

This kind of projection has been used since the earliest times in cartography under the name of stereographic projection. Its most important properties are the following:

B. To each point of the plane there corresponds one and only one point of the sphere, since each projector cuts the sphere in only one point besides the point O'.

C. Conversely, to each point of the sphere there corresponds one point of the plane. The point O' is apparently an exception; however, the theorem is made general by assuming as in previous paragraphs that the plane has one infinitely distant point and assigning this point to correspond to O'.

D. To each straight line of the plane there corresponds a circle of the sphere passing through O', and conversely,

(1) Burkhardt-Rasor, op. cit. pp. 47-[illegible] (with the exception of IV)

3. <u>Two such circles of the sphere intersect in the same angle as the two corresponding straight lines of the plane.</u>

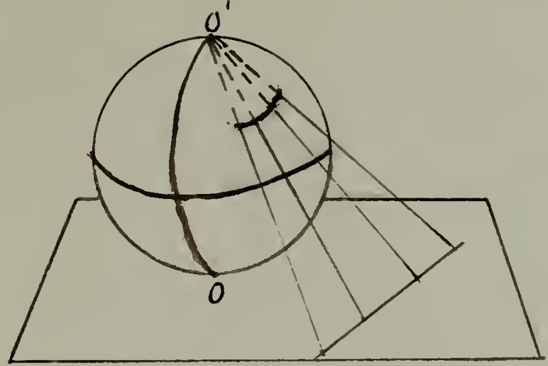


Fig. 6

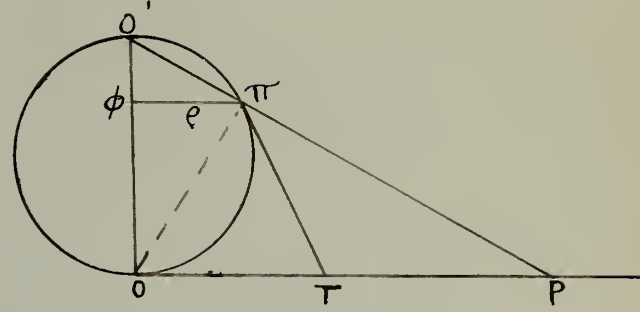


Fig. 7

To prove this theorem, let us pass the plane of reference, Fig. 7, through the vertices P π of both angles. The angle at P makes with O'P a solid angle. The planes tangent to the sphere at O and at π cut this solid angle in two angles, the first of which is the angle between the straight lines of the plane, while the second is equal to the angle between the corresponding circles on the sphere, since its sides are tangent to these circles. But both of these planes are normal to the plane of reference and their intersections πT, PT make with πP oppositely equal angles. (That is $\angle \pi PO = \angle \pi OO'$, each being complementary to the angle $\pi O'O$; and $\angle T\pi P = \angle O'O\pi$ being measured by half of the same arc $O'\pi$.) Moreover, the two tangent planes with reference to the edge of the solid angle are antiparallel (equally inclined to πP)(1) and cut it accordingly in the same angle(2). Hence, the two angles under comparison are equal.

(1) πT and PT are the projections of πP on the two tangent planes.
(2) If the intersection of the tangent planes is ATB, AB $\perp$ plane πTP. Rt.Δ ATπ $\cong$ rt.Δ ATP and rt.ΔBTπ $\cong$ rt.ΔBTP. Hence, ΔABπ $\cong$ ΔABP.

In this projection, points of the plane infinitely near each other are transformed into points indefinitely near each other on the sphere, and hence curves in the plane tangent to each other are transformed into curves tangent to each other on the sphere. Consequently, the following generalization of Theorem E is at once possible:

F. Any two curves of the sphere cut each other at each of their points of intersection in the same angle as the ~~same the~~ corresponding curves of the plane at the corresponding points of intersection.

We deduce further theorems with the aid of analytical geometry. We introduce the ξ, η, ζ rectangular space coordinates of which the ξ- and the η-axes coincide with the x- and y-axes of the $(x + iy)$-plane, while the positive direction of the ζ-axis is that of OO'. In this system of coordinates the equation of the sphere is

$$\xi^2+\eta^2 = \zeta(1-\zeta). \tag{1}$$

The point (ξ, η, ζ) of the sphere corresponds to the point of the plane whose coordinates are x, y and the radius vector $r = \sqrt{x^2+y^2}$. To obtain the ζ-coordinate of this point on the sphere and its distance ρ from the ζ-axis, the similar triangles $O'\phi\pi, \pi\phi O, O'OP$ in Fig. 7 furnish the following double proportion:

$$(1-\zeta):\rho = \rho:\zeta = 1:r.$$

From this it follows that

$$r = \frac{\zeta}{\rho} = \frac{\rho}{1-\zeta}, \tag{2}$$

and from these further:

$$r^2 = \frac{\zeta}{1-\zeta}\,; \qquad 1+r^2 = \frac{1}{1-\zeta}\,, \tag{3}$$

and conversely,

$$\zeta = \frac{r^2}{1+r^2}\,, \qquad \rho = \frac{r}{1+r^2}\,. \tag{4}$$

By construction it follows that

$$x : y : r = \xi : \eta : \rho\,.$$

We find therefore that

I. The coordinates of a point of the sphere are expressed as follows in terms of the coordinates of the corresponding point of the plane:

$$\xi = \frac{x}{1+r^2}\,, \quad \eta = \frac{y}{1+r^2}\,, \quad \zeta = \frac{r^2}{1+r^2}\,. \tag{5}$$

II. Conversely, the coordinates and radius vector of a point of the plane are expressed as follows in terms of the coordinates of the corresponding point of the sphere:

$$x = \frac{\xi}{1-\zeta}\,, \quad y = \frac{\eta}{1-\zeta}\,, \quad r^2 = \frac{\zeta}{1-\zeta}\,. \tag{6}$$

The following theorem is obtained at once from these formulas:

I. To any circle of the plane there corresponds a circle of the sphere, and conversely.

For, to the points of the plane satisfying the equation of the circle

$$a r^2 + b x + c y + d = 0, \tag{7}$$

there correspond the points of the sphere whose coordinates

satisfy the equation

$$a\zeta + b\xi + c\eta + d(1-\zeta) = 0. \tag{8}$$

But this is the equation of a plane and it cuts the sphere in a circle. This converse theorem, however, supposes the word "circle" (in the plane) to be taken in its extended sense to include the straight line.

We now transfer the geometrical representation of complex numbers from the plane to the sphere:

V. <u>We assign to each point of the sphere the same complex number $z = x + iy$ which heretofore belonged to its stereographic projection on the plane.</u>

Thus to the real numbers and the pure imaginaries, for example, there correspond on the sphere the points on the "meridians" $\eta = 0$ and $\xi = 0$ respectively; to the points of absolute value 1, there correspond the points on the "equator" $\zeta = \frac{1}{2}$. To opposite complex numbers (z and $-z$) correspond points of the sphere which are symmetrical to the ζ-axis, and to conjugate complex numbers ($x + iy$ and $x - iy$) correspond points symmetrical to the $\xi\zeta$-plane. To the number ∞ there corresponds on the sphere just as to any other complex number, one and only one point, viz. O'.

By means of this interpretation of complex numbers on the sphere we can now tell what transformation of the sphere into itself is represented by the function $z' = \frac{1}{z}$. Let (x, y) and (x', y') be two points of the plane which correspond to each

other by the transformation by reciprocal radii in reference to the unit circle; and let (ξ, η, ζ) and (ξ', η', ζ') be respectively their stereographic projections on the sphere. Then substitute in equations (6)(7) of part 1, the value of x, y, r^2 and x', y', r'^2 respectively from equations (8) of the present part and from the corresponding equations with accented letters, and we obtain:

$$\frac{\xi'}{1-\zeta'} = \frac{\xi}{\zeta}, \quad \frac{\eta'}{1-\zeta'} = \frac{\eta}{\zeta}, \quad \frac{\zeta'}{1-\zeta'} = \frac{1-\zeta}{\zeta},$$

from these it follows that

$$\xi' = \xi, \quad \eta' = \eta, \quad \zeta' - \tfrac{1}{2} = -\left(\zeta - \tfrac{1}{2}\right), \tag{9}$$

that is:

V. <u>The transformation by reciprocal radii in reference to the unit circle in the plane corresponds, by stereographic projection on the sphere, to a reflection on the equatorial plane $\zeta - \frac{1}{2} = 0$.</u>

The transformation of the plane into itself by means of $z' = z^{-1}$ is performed by first transforming by reciprocal radii in reference to the unit circle and then reflecting on the axis of real numbers. The corresponding transformation of the sphere into itself is thus performed by two reflections, one on the equatorial plane and the other on the meridian plane $\eta = 0$. Now these two reflections on the two planes perpendicular to each other are compounded by merely "reflecting on the line of intersection of the two planes", that is, by finding for each point another one symmetrical to the first in reference to the

line of intersection. This transformation is performed also by rotating the sphere through 180° about this line of intersection as an axis. Hence, we state the following theorem:

I. *The transformation $z' = z^{-1}$ determines a rotation of the sphere through 180° about the diameter passing through the points $z = 1$ and $z = -1$.*

In the plane the origin was an exception to the transformation in that there was no proper point corresponding to it. On the sphere, as we have seen, it is different, since the origin corresponds to its opposite pole O'. Hence we say

II. *The transformation $z' = z^{-1}$ is reversibly unique for all points of the sphere; to any point z there corresponds one and only one point z', and conversely.*

From the geometrical representation given in I we infer further that:

III. *For the transformation $z' = z^{-1}$ there are two and only two points z which coincide with their corresponding points z', viz. $z = 1$ and $z = -1$.*

Hence:

IV. *The stereographic projection of a plane on a sphere, and the stereographic projection of a sphere on a plane are conformal representations*, since they correspond to the conformal representation $z' = \frac{1}{z}$ in a plane.

3. Stereographic Projection of a Sphere on a Plane.(1) The sphere may be projected stereographically upon a plane as follows (see Fig. 8). Let the center of the sphere be taken as the origin of coordinates ξ, η, ζ of a point on the sphere. Let the points of the sphere be projected from the south pole (whose coordinates are 0, 0, -1) upon the tangent plane at the north pole and take the Cartesian axes Ox and Oy on the tangent plane parallel to the axes ξ and η, respectively. (The positive directions of x and y will correspond with those of ξ and η, respectively, since we are viewing the projection from below.)

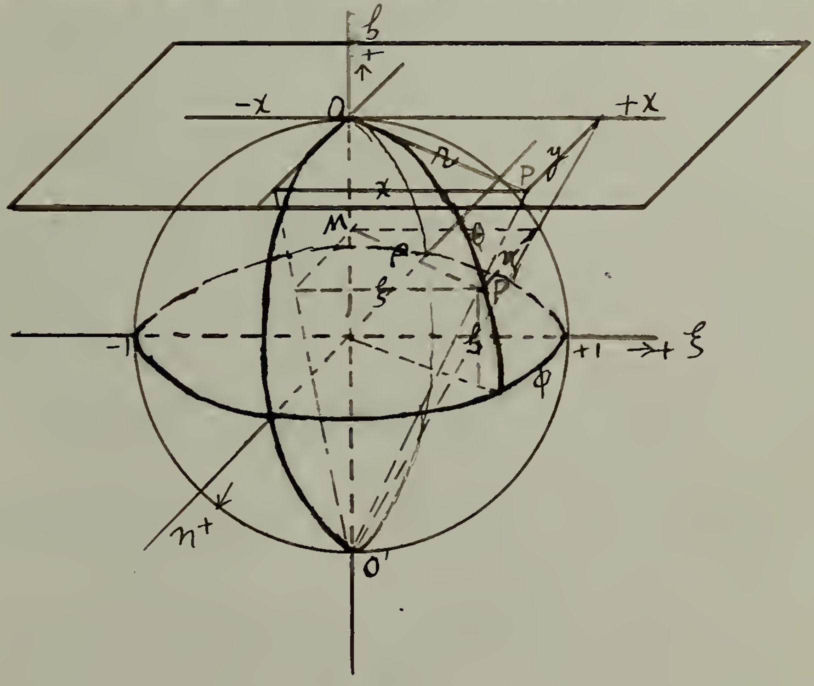


Fig. 8

Take a point $P'(\xi, \eta, \zeta)$ on the sphere and let its projection on the xy plane be $P(x, y)$. Let ϕ be the longitude of P' measured from the plane $\eta = 0$, and let θ be the north polar distance of P' (see Fig. 8).

(1) From a problem by [illegible]ardt-[illegible]er, op. cit. [illegible]

Pass a plane of reference (Fig. 8) through OO'P and P'.

Then

$$\frac{O'M}{\rho} = \frac{\rho}{OM} = \frac{OO'}{OP},$$

or
$$\frac{1+\zeta}{\rho} = \frac{\rho}{1-\zeta} = \frac{2}{r}.$$

Hence, $r = \dfrac{2\rho}{1+\zeta}$.

But, $x : y : r = \xi : \eta : \rho$.

Hence, $\dfrac{x}{\xi} = \dfrac{r}{\rho}$, and $\dfrac{y}{\eta} = \dfrac{r}{\rho}$,

and $x = \dfrac{r\,\xi}{\rho} = \dfrac{2\xi}{1+\zeta}$ and $y = \dfrac{\eta r}{\rho} = \dfrac{2\eta}{1+\zeta}$.

Hence, $x + iy = \dfrac{2\xi + 2i\eta}{1+\zeta} = \dfrac{2}{1+\zeta}(\xi + i\eta)$.

But, $\xi + i\eta = \rho(\cos\phi + i\sin\phi)$,

and $\rho = \cos(90° - \theta) = \sin\theta$,

and $\zeta = \sin(90° - \theta) = \cos\theta$.

$$\therefore x + iy = \frac{2\sin\theta}{1+\cos\theta}(\cos\phi + i\sin\phi)$$

$$= 2\frac{\sqrt{1-\cos^2\theta}}{1+\cos\theta}(\cos\phi + i\sin\phi)$$

$$= 2\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(\cos\phi + i\sin\phi)$$

$$= 2\tan\frac{\theta}{2}(\cos\phi + i\sin\phi).$$

$$\therefore \underline{P = 2\tan\frac{\theta}{2}(\cos\phi + i\sin\phi).}$$

On this map the meridians become a pencil of rays through the origin making $\angle\phi$ with ray $OX = 0$ and the parallels become circles with center at the origin and radius $= 2\tan\frac{\theta}{2}$. (See Fig. 9.).

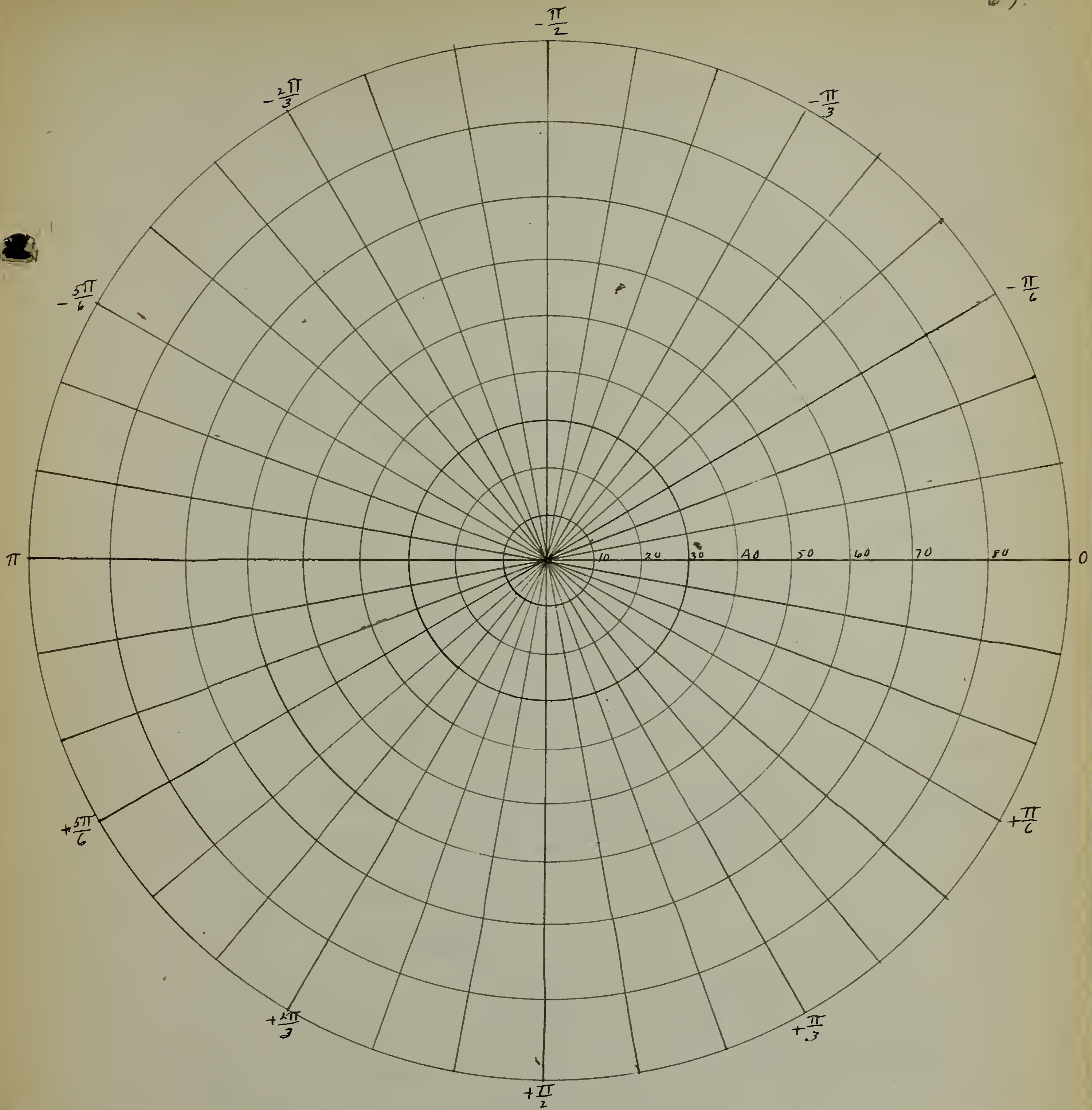


Fig. 9.

STEREOGRAPHIC PROJECTION

SCALE: 5 cm. = 1.

$P = 2\tan\frac{\theta}{2}(\cos\phi + i\sin\phi)$

θ is measured from N. pole.

ϕ is measured from plane $\eta = 0$.

4. <u>Conformal Representation Determined by the Logarithm.</u>[1]

We investigate now the conformal mapping of the z-plane upon the w-plane determined by the function

$$w = \log z. \tag{1}$$

In this connection we keep in mind the principal value of the logarithm. In the theory of the real logarithm of a real positive number $|z|$ it is known that such a logarithm takes on real values continually increasing from $-\infty$ to $+\infty$ as $|z|$ increases from 0 to ∞. Further, ϕ continually increasing passes from $-\pi$ to $+\pi$ as z describes a circle about the origin in the positive sense, starting at its intersection with the negative x axis and returning to that place. Since a circle about the origin and a radius vector starting at the origin can intersect in only one point it follows that:

I. <u>The principal value</u>

$$\underline{w = u + iv} \tag{2}$$

<u>of the logarithm takes on each finite complex value at one and only one point of the plane provided the imaginary part v satisfies the inequality:</u>

$$\underline{-\pi < v \leqq +\pi.} \tag{3}$$

But, expressed geometrically, this means that:

II. <u>The z-plane cut along the half axis of negative real numbers is mapped conformally by the principal value of the logarithm upon the parallel strip of the w-plane bounded by the lines $v = -\pi$ and $v = +\pi$.</u>

[1] Burkhardt-Rasor op. cit. pp. 304-305.

Thus the parallels to the u-axis correspond to the rays of the z-plane starting at the origin, the parallels to the v-axis correspond to the concentric circles about the origin in the z-plane.

Going now from the principal value to the other values of the logarithm, we find that:

C. The z-plane cut along the half-axis of negative real numbers is mapped by the kth value of the logarithm upon that strip of the w-plane bounded by the parallels:

$$v = (2k - 1)\pi, \quad v = (2k + 1)\pi.$$

The maps of the z-plane upon the w-plane determined by the different branches of the logarithmic function are therefore contiguous throughout the w-plane and finally cover the whole of it once without gaps. From this it follows that:

D. There is always one, and only one, value of z (finite and different from zero), for which one of the values of log z is equal to an arbitrary, preassigned finite complex number w.

5. Mercator's Projection.(1) The problem of making maps of the earth's surface by applying the principles of stereographic projection and conformal representation is of great interest. The discovery of the compass brought with it the idea of steering a course making with all the meridians a constant angle. This course was a spiral and was called a rhumb line or loxodrome. If the earth's surface (regarded as a sphere) be inverted from any point of the surface, say the north pole,

(1) From a problem, Burkhardt-Rasor, op. cit. pp. [illegible].

into a plane tangent at the south pole the meridians become a pencil of rays through the origin in the plane and the loxodromes are then, by isogonality, curves cutting this pencil at a constant angle, that is, equiangular spirals. But the map so formed by stereographic projection was not sufficiently simple since the loxodromes were the important lines. A map was wanted on which the loxodromes would appear as straight lines. This was accomplished by mapping the inverse of the sphere by means of $w = \log z$. And this, then, is the principle of Mercator's Projection.

If, in the stereographic projection defined in part C (see also Fig. 9) we introduce a new complex variable $w = u + iv = -i \log\left(\frac{z}{2}\right) = -i \log\left[\frac{1}{2}(x+iy)\right]$ we obtain another map of the surface of the sphere usually called Mercator's Projection.

Let $z = 2 \tan \frac{\theta}{2} (\cos\phi + i \sin\phi)$

and $w = u + iv = -i \log\left(\frac{z}{2}\right) = -i \log\left[\left(\frac{1}{2}\right)(x+iy)\right]$.

Then $w = -i \log\left[\left(\frac{1}{2}\right)\left(2 \tan\frac{\theta}{2}\right)(\cos\phi + i \sin\phi)\right]$

$$= -i \log\left[\left(\frac{1-\cos\theta}{1+\cos\theta}\right)^{\frac{1}{2}} \left(e^{i\phi}\right)\right]$$

$$= \log\left(\frac{1+\cos\theta}{1-\cos\theta}\right)^{\frac{i}{2}} + \log\left(e^{i\phi}\right)^{-i}$$

$$= i \log \cot\frac{\theta}{2} + \phi.$$

$\therefore \underline{u = \phi}$, and $\underline{v = \log \cot\frac{\theta}{2}}$.

When the sphere is mapped by this projection the parallels of latitude go into parallels to the u-axis at the distance $v = \log_e \cos\frac{\theta}{2}$ from the u-axis while the meridians go into parallels to the v-axis at the distance $u = \phi$ from the v-axis.

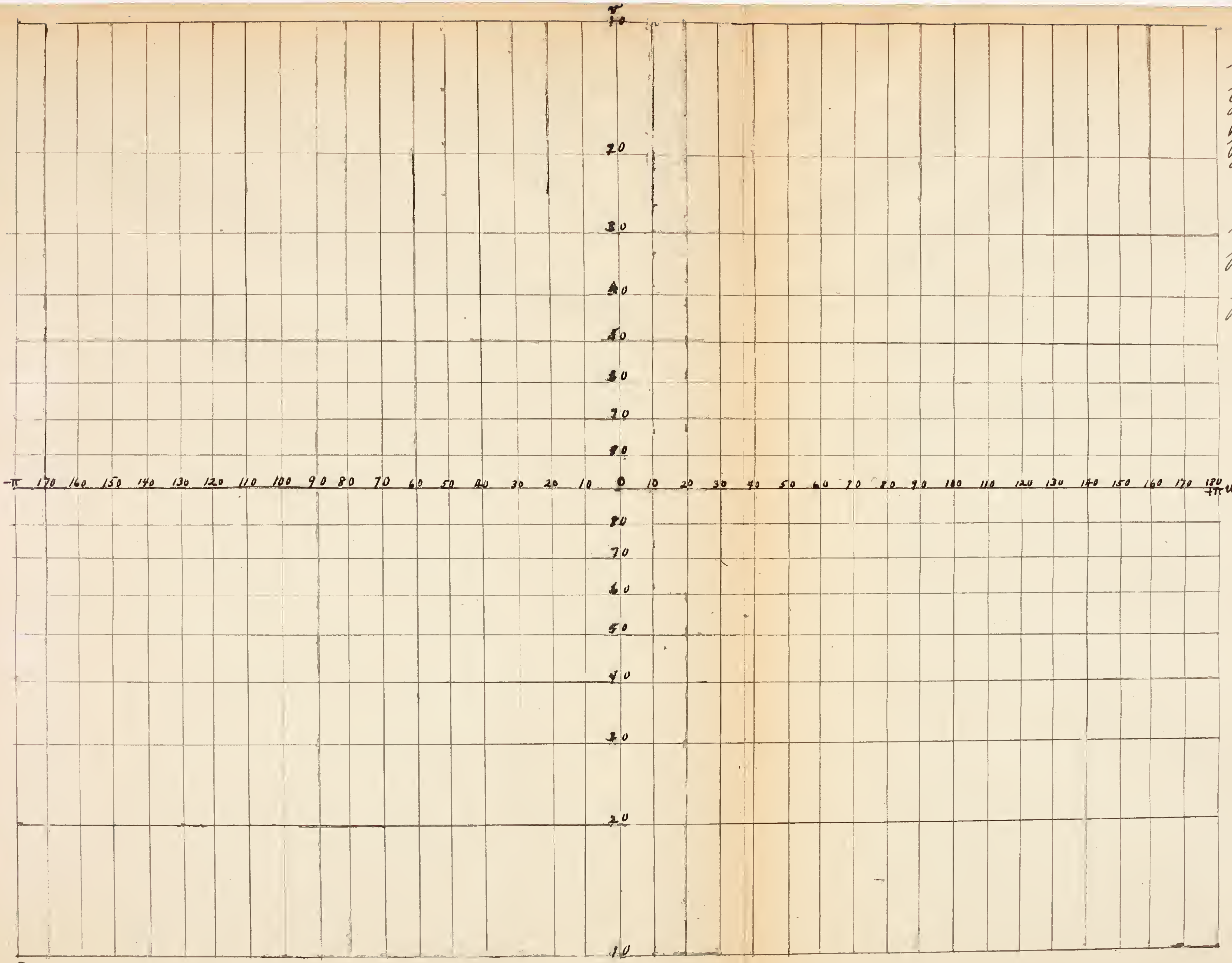


Fig. 10

MERCATOR'S PROJECTION — $w = u + iv = i \log_e \cot\frac{\theta}{2} + \phi$

SCALE — 5 cm. = 1.

$u = \phi,\ v = \log_e \cot\frac{\theta}{2}$

$(-\pi < u \leqq +\pi)$

ϕ is expressed in radians.

θ is measured from the poles.

$-\pi$ is on the next meridian beyond the last included in the map.

$+\pi$ is the last meridian on the map.

On this map parallels of latitude and the meridians are represented by straight lines parallel to the axes of u and v, respectively, (see Fig. 10). The u and v values of part 4 are interchanged by taking $w = -i \log z$, hence <u>we must place the following restriction on our map; $-\pi < u \leqq +\pi$ in order to have it singled valued.</u>

6. <u>Value of Mercator's and Stereographic Projections.</u>[1] We have already mentioned the fact that a rhumb line by Mercator's Projection is a straight line on the map. Hence, the map has great value for navigation. Its common use for world maps is very misleading since the scale increases as we go from the equator toward the poles. Hence, a scale of miles on a map of this type is impossible. The pole would be represented by a line $v = \infty$, hence the map is seldom extended beyond 80° of latitude.

Since a degree is 1/360 part of a circle, the degrees of latitude are everywhere equal on a sphere, as the meridians are all equal circles. The degrees of longitude, however, vary in the same proportion as the size of the parallels vary at the different latitudes. The parallel of 60° latitude is just one-half of the length of the equator. A square-degree quadrangle at 60° of latitude has the same length north and south as has such a quadrangle at the Equator, but the extent east and west is just one-half as great. Its area, then is approximately one-half the area of the one at the Equator. Now, on

[1] Dietz and Adams, Elements of Map Projection, pp. [illegible].

Mercator's Projection the longitude at 60° is stretched to double its length, and hence the scale along the meridian has to be increased an equal amount. The area is therefore increased fourfold. At 80° of latitude the area is increased to 33 times its real size, and at 85° an area would be more than 130 times as large as an equal sized area at the Equator. This excessive exaggeration of area is a most serious matter if the map be used for general purposes. For instance, Greenland shows larger than South America, but South America is nine times as large as Greenland.

We must not lose sight, however, of the fact that by this projection the whole inhabited world can be shown on one sheet. Also by letting $u > \pi$ (or $< -\pi$) we can extend the map to either the east or west.

A stereographic projection is well adapted for mapping the polar regions. As we recede from the pole of the tangent plane to that of the point of projection the magnification increases but slightly for regions near the pole. At 30° from the pole the increase is less that 10%, at 70° it is less than 50%, but at 90°, or the equator, the scale is about twice that of regions near the pole. After the equator is crossed the magnification increases too rapidly for practical purposes becoming infinite at the south pole.

VII Summary

The diversity of the topics covered has made it quite evident that the use of imaginaries as an instrument in solution is not limited to any one branch of mathematics, but has a wide application.

DeMoivre's Theorem, and Its Applications

Assuming an elementary knowledge of the representation of complex numbers, and operations with them, we started with DeMoivre's theorem which states that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. We then made several applications of this analytical tool.

First we showed by use of the binomial theorem that cosine $n\theta$ and sine $n\theta$ can be expressed in terms of cosine θ and sine θ as follows:

$$\cos n\theta = c^n - \frac{n(n-1)}{1\cdot 2}c^{n-2}s^2 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}c^{n-4}s^4 - \cdots \text{ etc.},$$

$$\text{and } \sin n\theta = nc^{n-1}s - \frac{n(n-1)(n-2)}{1.2.3}c^{n-3}s^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}c^{n-5}s^5 \ldots \text{ etc.},$$

where $c = \cos\theta$ and $s = \sin\theta$.

We then developed by the use of the binomial expansion the following expressions for cosine θ and sine θ in terms of multiple angles:

$$2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{1\cdot 2}\cos(n-4)\theta + \cdots, \text{etc.},$$

and $2^{n-1}i^n\sin^n\theta = \cos n\theta - n\cos(n-2)\theta + \frac{n(n-1)}{1\cdot 2}\cos(n-4)\theta - \cdots, \text{etc.},$

when n is even, and

$$= i\left[\sin n\theta - n\sin(n-2)\theta + \frac{n(n-1)}{1\cdot 2}\sin(n-4)\theta - \cdots, \text{etc.}\right],$$

when n is odd.

The imaginary factor disappears in each of these cases.

By writing $\sqrt[n]{z}$ as $r^{\frac{1}{n}}(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}) = r^{\frac{1}{n}}(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n})$, where k is any integer and then showing that the substitution of any number of values for k gives only n distinct values of $\sqrt[n]{z}$, we showed that every complex number, $z = r(\cos\theta + i\sin\theta)$ has n, and only n roots which are given by the formula

$$\left[r(\cos\theta + i\sin\theta)\right]^{\frac{1}{n}} = \left[\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right],$$

where $k = 0, 1, 2, \ldots, n-1$.

By writing the n roots of 1 as shown in the preceding paragraph we showed that the n roots of the equation, $z^n - 1 = 0$ are represented geometrically by the n lines (or their terminal points) drawn from the origin as center to a circle, radius unity, with one of the lines coinciding with the positive direction of the x-axis, so that the circle is divided into n equal parts.

In a similar manner, using -1 instead of 1, we showed that the n roots of the equation $z^n+1=0$ are represented geometrically by the n lines (or their terminal points) drawn from the origin as center to the circle, radius unity, the positive x-axis being taken to bisect the angle between a pair of consecutive lines, so that the circle is divided into n equal parts.

Our next section dealt with the character of the cube roots of unity. We showed that the square of either of the imaginary cube roots of unity equals the other, and their product equals unity.

We then extended our discussion to show that any two roots of a number may be obtained by multiplying the third by ω and ω^2, respectively, where $\omega = -1 \pm \frac{i\sqrt{3}}{2}$.

By reducing the general cubic equation, $a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$ to the form, $z^3 + 3Hz + G = 0$ and letting $z = u + v$, we showed by analysis that the three roots can be expressed as

$$z_0 = u - \frac{H}{u},$$
$$z_1 = \omega u - \frac{H}{\omega u} = \omega u - \frac{\omega^2 H}{u},$$
$$z_2 = \omega^2 u - \frac{H}{\omega^2 u} = \omega^2 u - \frac{\omega H}{u},$$

where u is one of the cube roots of $\frac{-G+\sqrt{G^2+4H^3}}{2}$ and ω has the value given above. x in this case equals $\frac{z-a_1}{a_0}$.

When G^2+4H^3 is negative, its square root is imaginary, and we have the so-called 'irreducible case'. We showed that its solution can be effected by the use of DeMoivre's theorem, which gives the roots as

$$z_0 = u - \frac{H}{u} = 2\sqrt{-H}\cos\frac{\theta}{3},$$
$$z_1 = \omega u - \frac{H}{\omega u} = 2\sqrt{-H}\cos\frac{\theta+2\pi}{3},$$
$$z_2 = \omega^2 u - \frac{H}{\omega^2 u} = 2\sqrt{-H}\cos\frac{\theta+4\pi}{3},$$

from which $x = \frac{z-a_1}{a_0}$.

We closed this part of our work with a discussion of the possibility of the construction of a regular polygon of 17 sides by the use of ruler and compasses.

For this purpose we made use of the fact which we had previously shown that the division of the circle into seventeen equal parts is equivalent to the solution of the equation $x^{17}-1=0$. We showed by an elaborate analysis that the trigonometric form of the first and sixteenth roots can be readily constructed by the erection of a single perpendicular provided we can first

construct a line which equals

$$\frac{-1+\sqrt{17}+\sqrt{34-2\sqrt{17}}}{8}+\frac{\sqrt{68+12\sqrt{17}-16\sqrt{34+2\sqrt{17}}-2(1-\sqrt{17})\sqrt{34-2\sqrt{17}}}}{8}$$

This line can be constructed by the methods of plane geometry since it can be derived from the known quantities by a finite number of rational operations and square roots.

Euler's Theorem, $e^{i\theta} = \cos\theta + i\sin\theta$.

We demonstrated the truth of Euler's theorem, $e^{i\theta} = \cos\theta + i\sin\theta$ by showing that $e^{i\theta}$ as represented by infinite series equals the sum of the two series which represent $\cos\theta$ and $i\sin\theta$, respectively.

We established the same result by writing DeMoivre's theorem as $(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n})^n$, and letting n increase without limit. By the theory of limits we showed that the limit $\cos\theta + i\sin\theta$ equals the limit $e^{i\theta}$.

We showed by a geometrical representation that the limit of $(1+\frac{i\theta}{n})^n$ as n increases without limit is $\cos\theta + i\sin\theta$. Since we had previously shown that the limit of $(1+\frac{i\theta}{n})^n$, as n increases without limit, approaches the same limit as $(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n})$ as n increases without limit, this figure demonstrated that $(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n})^n$ approaches $\cos\theta + i\sin\theta$ as n increases without limit.

From Euler's theorem we readily obtained that $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$, relations which can be shown to hold for any value of θ, whether real or complex.

By assuming a knowledge of imaginaries and exponentials, and defining the sine and cosine as above, we showed how the whole of trigonometry can be made an easy application of algebra.

Hyperbolic Functions

We replaced θ in Euler's theorem by $i\theta$ and $-i\theta$ in turn. By taking half the sum and half the difference of these two results we obtained respectively, that $\cos i\theta = \frac{e^{\theta} - e^{-\theta}}{2}$ and $-i \sin i\theta = \frac{e^{\theta} - e^{-\theta}}{2}$ which expressions we defined respectively as the hyperbolic cosine and the hyperbolic sine of θ. The hyperbolic tangent was defined as the ratio of the hyperbolic sine to the hyperbolic cosine, and the reciprocals of the hyperbolic sine, hyperbolic cosine and hyperbolic tangent were defined respectively, as the hyperbolic secant, hyperbolic cosecant and hyperbolic cotangent.

We stated that there existed a corresponding property of or relation between hyperbolic functions for every property of or relation between circular functions, and showed how these properties and relations can be derived by the use of exponentials, or by the substitution of $i\theta$ for θ in the formulas of circular functions.

We showed that each of the hyperbolic functions has a pure imaginary period of $2i\pi$, and the hyperbolic tangent and cotangent have the smaller period of $i\pi$.

We then showed that the hyperbolic functions can be expressed as ratios of certain lines connected with the equilateral hyperbola, just as the circular functions can be expressed as rations of certain lines connected with the circle.

To obtain the inverse hyperbolic cosine of y, $x = \cosh^{-1} y$, where $\cosh x = y = \frac{e^{x} + e^{-x}}{2}$, we solved this last equation for x.

Our result shows that $x = \cosh^{-1} y = \log(y \pm \sqrt{y^2 - 1})$. In a similar manner we obtained the following results:

$$\sinh^{-1} y = \log(y + \sqrt{y^2 + 1}), \quad \tanh^{-1} y = \tfrac{1}{2} \log \frac{1+y}{1-y}, \quad \coth^{-1} y = \tfrac{1}{2} \log \frac{y+1}{y-1},$$

$$\operatorname{sech}^{-1} y = \log \frac{1 + \sqrt{1 - y^2}}{y}, \text{ and } \operatorname{csch}^{-1} y = \log \frac{1 + \sqrt{1 + y^2}}{},$$

From the periodicity of hyperbolic functions it follows that the inverse functions are multiple valued. The particular value of $\sinh^{-1} y$ which lies between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, of $\cosh^{-1} y$ which lies between 0 and π, and of $\tanh^{-1} y$, which lies between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ is called the principal value of $\sinh^{-1} y$, $\cosh^{-1} y$ and $\tanh^{-1} y$, respectively.

As an interesting case of the analogy of the hyperbolic functions and circular functions we show how the roots of a cubic equation, when one root is real and two are complex, can be expressed by hyperbolic functions in a form analogous to that by which the roots of a cubic equation, when all their roots are real, are expressed by circular function. Our formulas are as follows. When all their roots are real we have

$$-2b^{\frac{1}{2}} \sin \frac{n}{3}, \quad b^{\frac{1}{2}}\left(\sin \frac{n}{2} \pm \sqrt{3} \cos \frac{n}{3}\right), \text{ where } n = \sin^{-1} \frac{c}{b^{3/2}}.$$

When one root is real and two are complex, we have either

$$2b^{\frac{1}{2}} \cosh \frac{n}{3}, \quad -b^{\frac{1}{2}}\left(\cosh \frac{n}{3} \pm i\sqrt{3} \sinh \frac{n}{3}\right), \text{ where } n = \cosh^{-1} \frac{c}{b^{3/2}},$$

or

$$2b^{\frac{1}{2}} \cosh \frac{n}{3}, \quad -b^{\frac{1}{2}}\left(\sinh \frac{n}{3} \pm i\sqrt{3} \cosh \frac{n}{3}\right), \text{ where } n = \sinh^{-1} \frac{c}{b^{3/2}},$$

according as the coefficient of the x term is negative or positive.

We closed our discussion of hyperbolics with a brief mention of their use in applied mathematics.

Use of Complex Numbers in the Solution of Several Isolated Problems

By writing $a^2+b^2+c^2+d^2$ and $x^2+y^2+z^2+w^2$ as determinants in which the elements were complex numbers we showed that the product of the two determinants can be expressed as a single determinant which represents the sum of four other squares, $m^2+n^2+p^2+q^2$, or, expressed in words, that a product of two factors each of which is the sum of four squares is itself the sum of four squares.

We then showed that we can determine the partial fraction corresponding to a factor x^2+1 in the denominator by substituting i for x after the equation with undetermined coefficients has been cleared of fractions and is in the form, $F(x) = (Ax+B)(x-a)(x-b)^2 + Q(x^2+1)$. A pair of simultaneous equations will be formed from which A and B can be readily found.

Easy solutions for the integrals, $\int e^{ax} \sin bx\, dx$ and $\int e^{ax} \cos bx\, dx$ were found by the use of exponential forms for $\sin bx$ and $\cos bx$.

In the case of the solution of the linear differential equation with constant coefficients,

$$k_0 \frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + k_{n-1} \frac{dy}{dx} + k_n y = X,$$

if some, or all of the roots of the auxiliary equation

$$k_0 m^n + k_1 m^{n-1} + \cdots\cdots + k_{n-1} m + k_n = 0$$

are complex, we showed that by grouping these roots as pairs

of conjugate complex numbers and substituting trigonometric forms for the exponentials in the resulting terms of the complementary function and $\frac{A-iB}{2}$ and $\frac{A+iB}{2}$ for the two constants in the same, we can change these complex terms to real terms. Each pair will then take the general form, $e^{\alpha x}(A\cos\beta x + B\sin\beta x)$ where A and B are the two arbitrary constants.

In the case of the solution of the homogeneous linear partial differential equation with constant coefficients, if we arrange the complex roots by pairs of conjugate complex numbers then by substituting $\phi_1 + i\psi_1$ for ϕ, and $\phi_1 - i\psi_1$ for ψ, in the corresponding terms of $\phi(y+\alpha x+i\beta x) + \psi(y+\alpha x-i\beta x)$ of the complementary function, the resulting expression, $\phi_1(y+\alpha x+i\beta x) + \phi_1(y+\alpha x-i\beta x) + i[\psi_1(y+\alpha x+i\beta x) - \psi_1(y+\alpha x-i\beta x)]$ will be real, when ϕ_1 and ψ_1 are any real functions.

Conformal Representation, and [illegible]

We first showed that the transformation $z' = \frac{1}{z}$ is equivalent to two simpler geometric transformations, (1) $\bar{x} = x$, $\bar{y} = -y$, and (2) $x' = \frac{\bar{x}}{\bar{x}^2+\bar{y}^2}$, $y' = \frac{\bar{y}}{\bar{x}^2+\bar{y}^2}$, each of which gives a conformal representation with inversion of angles. Hence, the result of the two transformations is a conformal representation without inversion of angles.

We then showed with the aid of solid analytic geometry that if the points of a plane are projected on a sphere by stereographic projection, then the transformation $z' = \frac{1}{z}$ in a plane corresponds to a rotation the sphere through 180° about an axis passing through the center and perpendicular to both the other

two axes of reference which have their origin at the center, and conversely.

The stereographic projection of a plane on a sphere, or of a sphere on a plane are conformal representations which correspond to the conformal representation $z' = \frac{1}{z}$ in a plane with a point to point correspondence.

We then showed how a sphere can be represented conformally by a stereographic projection from the south pole (coordinates 0, 0, -1) upon a plane tangent at the north pole. The points of our map fulfil the equation $Z = 2 \tan\frac{\theta}{2}(\cos\phi + i \sin\phi)$.

We then showed that the conformal mapping of the z-plane on the w-plane determined by the function $w = \log z$ will give a point to point correspondence for the principal values of the logarithm provided the imaginary part v satisfies the inequality $-\pi < v \leqq +\pi$. This statement, expressed geometrically, means that the z-plane cut along the half-axis of negative real numbers is mapped conformally by the principal value of the logarithm upon the parallel strip of the w plane bounded by the lines $v = -\pi$, and $v = +\pi$. For each of the other values of the logarithms there is a similar strip parallel to the one mentioned above on which the map is repeated. If all the values of the logarithm are considered the entire w-plane would be used, since the strips are contiguous.

We then showed how we can transform our map of the sphere formed by stereographic projection into a conformal representation in which rhumb lines will be straight by the so-called Mercator's projection. Our equation for this change is $w = -i \log\left(\frac{Z}{2}\right)$.

The Mercator's Projection has great value for navigation, since a rhumb line is straight, but its use for world maps is misleading since the magnification increases as we go from the equator. We can, however, represent the whole inhabited world on one sheet, and can extend it to either the east or west, as desired. The stereographic projection is particularly adapted for mapping the polar regions. Because of the magnification it is not practical to extend such a map beyond the equator.

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